THE MATHEMATICS STUDENT

EDITED BY

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WITH THE CO-OPERATION OF

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ERRATUM

p. 123, line 2. *For* 'would be called' *read* 'would not be called'

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ON INVARIANT SPACES OF A MATRIX

By A. C. CHOUDHURY

1. The connection between the invariant subspaces of a given linear transformation A of a vector space V over a field F into itself and the ideals in the polynomial domain $F[x]$ is well known [3]. But the characterization of the lattice structure in the set of invariant subspaces is not known. It will be proved in this note that the lattice of invariant subspaces is dual to the lattice of certain polynomial ideals. One can proceed further and prove that the lattice of invariant subspaces is isomorphic to the lattice of factorization of the minimal polynomial $m(x)$. This last characterization gives at once the two decomposition theorems of a matrix.

2. Represent the linear transformation by a matrix A. Establish a binary relation R between the polynomials of $F[x]$ and the vectors of V by the condition that $p(x)\ R\ \alpha$ if and only if $p(A)\ \alpha = 0$. This binary relation R determines two closures, one in $F[x]$ and the other in V. A closed set in $F[x]$ is an ideal I_p formed by the multiples of a polynomial $p(x)$ and a closed set in V is the solution space of the homogenous linear equations $p(A)\ X = 0$. The solution space may be denoted by V_p, and I_p is the annihilator of this vector space V_p. One can express this relation between them in the form

$$V_p = R(I_p).$$

The vector space V_p is an invariant one, as

$$p(A)\,(A\alpha) = A(p(A)\,\alpha) = 0$$

if $\alpha \in V_p$ and thus $A\alpha \in V_p$ if $\alpha \in V_p$. Then it follows from the general theory of the closures generated by the binary relations, that the subspaces V_p form a lattice with V as the maximum element and the ideals I_p form another lattice having I_m as the minimum ideal and further that $V_p = R(I_p)$ is a duality between these two lattices. Hence

(1) $V = V_m$, where $m(x)$ is called the minimal polynomial;

(2) $R(I_{p_1} \cap I_{p_2}) = V_{p_1} + V_{p_2}$;

(3) $R(I_{p_1} + I_{p_2}) = V_{p_1} \cap V_{p_2}$;

(4) if $I_{p_1} \subseteq I_{p_2}$, then $V_{p_1} \supseteq V_{p_2}$;

where in both lattices, the meet of two sets is the lattice product and the sum is denoted by the symbol + and consequently the set inclusion is the lattice inclusion.

3. The lattice of invariant subspaces has another characterization. It can be represented by the lattice of factorization of the minimal polynomial $m(x)$.

The non-zero elements of an integral domain with unique factorization form a lattice with the h.c.f. as the lattice product and the l.c.m. as the lattice sum. In this lattice, the inclusion relation is given by the condition that $q \subseteq p$ if and only if q is a factor of p and, hence, this lattice is called the lattice of divisibility or the lattice of factorization. As F is a field, the polynomial domain is Euclidean and hence the factorization is unique in $F[x]$. Thus the lattice of factorization is defined for the domain $F[x]$. Now as I_{p_0} is the ideal formed by the multiples of $p(x)$, the correspondence $p \longleftrightarrow I_p$ is a dual isomorphism between the lattice of divisibility and the lattice of ideals of $F[x]$, i.e.

$$I_p \subseteq I_q \longleftrightarrow q \leqslant p.$$

In particular, the factors of the minimal polynominal $m(x)$ form a sublattice dually isomorphic to the lattice of ideals I_p such that $F[x] \supseteq I_p \supseteq I_m$. Consequently, the invariant subspaces of the linear transformation A form a lattice isomorphic to the lattice of factors of the minimal polynomial $m(x)$. Thus

(1) $V_{[pq]} = V_p + V_q$;

(2) $V_{(pq)} = V_p \cap V_q$;

(3) if p is a factor of q, $V_p \subseteq V_q$;

where (pq) is the h.c.f. and $[pq]$ is the l.c.m. of p and q.

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A NOTE ON A THEOREM OF BOREL

By MANSOOR AHMAD

1. Introduction. The object of this note is to generalize a well-known theorem of Borel, such that the essential idea of the theorem is preserved, though in a modified form which is closely related to the theorem.

2. Lemmas. (i) *If* (a_m) *and* (b_m), $m = 1, 2, 3, \ldots$ *are two sequences of positive numbers, tending to infinity with* m *such that* $a_m \leqslant b_m$ *for all sufficiently large* m, *then*

$$\text{(A)} \liminf_{m \to \infty} \frac{a_m}{b_m} \leqslant \liminf_{m \to \infty} \frac{l_k a_m}{l_k b_m},$$

$$\text{(B)} \limsup_{m \to \infty} \frac{a_m}{b_m} \leqslant \limsup_{m \to \infty} \frac{l_k a_m}{l_k b_m}.$$

(ii) *If* (a_m) *and* (b_m) *are two sequences of positive numbers, tending to infinity with* m, *such that either* (A) *or* (B) *holds with the inequality sign, then* $a_m < b_m$ *for an infinity of* m; *here* $l_k x$ *denotes the* k*-th iterate of* $\log x$.

PROOF OF (i) (A). Let us suppose that

$$\liminf_{m \to \infty} \frac{a_m}{b_m} > \liminf_{m \to \infty} \frac{\log a_m}{\log b_m} = p. \tag{1}$$

Then we have

$$a_m > (p + \epsilon)\, b_m$$

for all $m > m_0$. Consequently, we have

$$\liminf_{m \to \infty} \frac{\log a_m}{\log b_m} = 1. \tag{2}$$

Combining (1) and (2), we have

$$\liminf_{m \to \infty} \frac{a_m}{b_m} > 1.$$

But, under our hypothesis, this is untenable; and hence

$$\liminf_{m\to\infty} \frac{a_m}{b_m} \leqslant \liminf_{m\to\infty} \frac{\log a_m}{\log b_m}$$

Similarly, we can show that

$$\liminf_{m\to\infty} \frac{\log a_m}{\log b_m} \leqslant \liminf_{m\to\infty} \frac{\log\log a_m}{\log\log b_m}$$

and so on. This completes the proof of (i) (A).

The proof of (i) (B) is similar, and (ii) follows easily from (i).

3. (i). Let $(r_m), m = 1, 2, 3, \ldots$ be a sequence of positive numbers, tending to infinity with m, and let $f(z)$ be a meromorphic function. We define α as a value e.k.m.l. for $f(z)$ if

$$\Delta_{k,m}(\alpha) = 1 - \liminf_{m\to\infty} \frac{l_k N(r_m, \alpha)}{l_k T(r_m)} > 0.$$

Similarly, we say that a value α is e.k.m.u. for $f(z)$ if

$$\delta_{k,m}(\alpha) = 1 - \limsup_{m\to\infty} \frac{l_k N(r_m, \alpha)}{l_k T(r_m)} > 0.$$

(ii). If $f(z)$ is a meromorphic function of finite kth order, and if the two limits

$$\rho_1(m, \alpha) = \lim_{m\to\infty} \frac{l_k n(r_m, \alpha)}{\log r_m}$$

$$\rho(m) = \lim_{m\to\infty} \frac{l_k T(r_m)}{\log r_m}$$

exist and satisfy the inquality

$$\rho_1(m, \alpha) < \rho(m),$$

then α is said to be v.e.m. for $f(z)$, where the kth order of $f(z)$ is defined as

$$\rho_k = \limsup_{r\to\infty} \frac{l_k T(r)}{\log r}.$$

THEOREM. (i). *If $f(z)$ is a meromorphic function and if α is a* v.e. k_1.m.l. *for $f(z)$, then α is a* v.e.k.m.l. *for $f(z)$ and $\Delta_{k,m}(\alpha) = 1$, when* $0 \leqslant k < k_1$.

(ii). *If $f(z)$ is a meromorphic function and if α is a* v.e.k$_1$.m.u. *for $f(z)$, then α is a* v.e.k.m.u. *$f(z)$ and $\delta_{k,m}(\alpha) = 1$, when $0 \leqslant k < k_1$.*

(iii). *If $f(z)$ is an entire function of finite k-th order but of infinite $(k-1)$-th lower order, there exists at most one* v.e.m. *for $f(z)$.*

(iv). *If $f(z)$ is a meromorphic function of finite k-th order but of infinite $(k-1)$-th lower order, there exist at most two values* e.m. *for $f(z)$, where the k-th lower order of $f(z)$ is defined as*

$$\lambda_k = \liminf_{r\to\infty} \frac{l_k T(r)}{\log r}.$$

(v). *If $f(z)$ is a meromorphic function of k-th order and if α is a* v.e.B *(in the sense of Borel) for $f(z)$ it is also a* v.e.m. *for a certain sequence (r_m), $m = 1, 2, 3, \ldots$; but the converse is not true.*

(i) and (ii) follow easily from the lemma. We prove (iii) and the proof of (iv) is similar. To show that the second part of (v) holds, we can consider the function

$$f(z) = e^{Q(z)} P(z),$$

where $P(z)$ is canonical product formed with the simple zeros (z_n), $n = 1, 2, 3, \ldots$, $|z_n| = n^{\beta_n}$,

$$\beta_n = 1 + \frac{1}{\alpha} + \cos\left\{\frac{\pi}{\log p} \log\left(\frac{\log n}{\log 2}\right)\right\},$$

α being any fixed positive integer and $Q(z)$ any polynomial of degree α. Here $r_m = m^{\beta_m}$, $m = 2^{p^{2k}}$, p being any fixed positive integer greater than $e^{\pi\alpha}$; and $k = 1, 2, 3, \ldots$.

PROOF OF (iii). By the second fundamental theorem of Nevanlinna [1, § 34], we have

$$T(r, f) = T(r) < N(r, \alpha) + N(r, \beta) + 8 \log T(cr) + O(\log r),$$

when $r > r_0$. It can be easily seen, that under our hypothesis,

$$\lim_{r\to\infty} \frac{\log T(cr)}{T(r)} = 0.$$

Consequently, we have

$$l_k\{T(r) - 8 \log T(cr) - O(\log r)\} \leqslant l_k\{N(r, \alpha) + N(r, \beta)\}.$$

Therefore

$$\limsup_{r\to\infty} \frac{l_k T(r)}{l_k\{n(r, \alpha) + n(r, \beta)\}} \leqslant 1. \tag{3}$$

Now, let us suppose that there exist two values α and β e.m. for $f(z)$. We have

$$n(r_m, \alpha) < e_{k-1} (l_{k-1} T(r_m))^{1-\epsilon}$$

and

$$n(r_m, \beta) < e_{k-1} (l_{k-1} T(r_m))^{1-\epsilon}$$

for all $m > m_0$, where $e_1(x) = e^x$, $e_2(x) = e^{e^x}$ and so on. Therefore, by (3), we have

$$\lim_{m\to\infty} \frac{l_k T(r_m)}{l_k\{2\, e_{k-1}(l_{k-1} T(r_m))^{1-\epsilon}\}} \leqslant 1$$

or

$$\lim_{m\to\infty} \frac{l_k T(r_m)}{(1-\epsilon)\, l_k T(r_m)} \leqslant 1,$$

which is a contradiction; and thus the theorem is proved.

Note. From the second fundamental theorem of Nevanlinna, loc. cit., we can easily deduce that

$$T(r, f) = T(r) < N(r, f - f_1) + N(r, f - f_2) + 8 \log T(cr) + \\ + a\, T(r, f_1) + b\, T(r, f_2) + O(\log r)$$

for all sufficiently large r, where $f(z)$, $f_1(z)$ and $f_2(z)$ are entire functions, a and b are certain positive constants, $n(r, f - f_1)$ denotes the number of zeros of $f(z) - f_1(z)$ in the region $|z| \leqslant r$, and $N(r, f - f_1)$ refers to $n(r, f - f_1)$. Using this form of the theorem, we can prove the following form of Theorem (iii):

If $f(z)$ is an entire function of finite k-th order but of infinite $(k-1)$-th lower order, there exists at most one function $f_1(z)$ of finite $(k-1)$-th order, which is e.m. *for $f(z)$.*

A similar generalized form of Theorem (iv) can be easily proved by the same method.

Finally, I must thank the U.P. Government for the monetary help given to me.

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ON THE DERIVATIVES OF INTEGRAL FUNCTIONS

By R. P. SRIVASTAV*

1. IF $f(z)$ is an integral function of order ρ and lower order λ and $f^{(s)}(z)$ is the sth derivative of $f(z)$ and $M(r)$, $M^{(1)}(r), \ldots,$ denote the maximum moduli of $f(z)$, $f^{(1)}(z), \ldots,$ respectively on the circle $|z| = r$, then [1]

THEOREM A. *The sequence* $M(r)$, $M^{(1)}(r), \ldots, M^{(s)}(r) \ldots,$ *where* $r \geqslant r_0 = r_0(f)$, *forms an increasing sequence provided that* $f(z)$ *is an integral function of lower order* λ, $\lambda > 1$.

THEOREM B. *The sequence* $M(r)$, $M^{(1)}(r) \ldots, M^{(s)}(r) \ldots,$ *where* $r \geqslant r_0 = r_0(f)$, *forms a decreasing sequence provided that* $f(z)$ *is of order less than* $\frac{1}{2}$.

Shah [2] has extended this result for $\rho < 1$.

THEOREM C. *For* $\lambda > 1$,

$$M^{(s)}(r) > M(r)\left[\frac{\log M(r)}{r \log r}\right]^s, \quad r \geqslant r_0 = r_0(f).$$

The object of this paper is to obtain certain properties of the maximum moduli of the derivatives of integral functions. The results have been given in the form of six theorems based on the above theorems of Dr. S. K. Bose and certain results of Dr. S. M. Shah.

2. THEOREM 1. *If for* $r \geqslant r_0 = r_0(f) > 1$, $M(r)$, $M^{(1)}(r), \ldots$ *form a non-increasing sequence, then the lower order* λ *of* $f(z)$ *is less than* 1.

PROOF. From Vijayaraghavan's [3] inequality it easily follows that

$$\liminf_{r\to\infty}\left[\frac{\log\{rM^{(1)}(r)/M(r)\}}{\log r}\right] \geqslant \lambda; \tag{2.1}$$

* I wish to thank Dr. S. K. Bose for suggesting this problem and for his guidance.

therefore

$$\frac{M^{(1)}(r)}{M(r)} > r^{\lambda-\epsilon-1}, r \geqslant r_0 > 1.$$

Since $M^{(1)}(r) \leqslant M(r)$, the result follows.

3. THEOREM 2. *If $M(r)$, $M^{(1)}(r), \ldots M^{(s)}(r) \ldots$, form a nondecreasing sequence, $\rho > 1$. ($r \geqslant r_0 = r_0(f) > 1$).*

PROOF. Shah [2] has shown that

$$\limsup_{r\to\infty}\left[\frac{\log\{r\, M^{(1)}(r)/M(r)\}}{\log r}\right] = \rho. \tag{3.1}$$

This leads to

$$\frac{M^{(1)}(r)}{M(r)} < r^{\rho+\epsilon-1}, \text{ for } r \geqslant r_0$$

or

$$1 < \frac{M^{(1)}(r)}{M(r)} < r^{\rho+\epsilon-1}, (r \geqslant r_0 > 1),$$

and hence $\rho > 1$.

4. THEOREM 3. *If $f(z)$ is an integral function of regular growth, then*

$$\lim_{r\to\infty}\left[\frac{\log r\ \{M^{(s)}(r)/M(r)\}^{1/s}}{\log r}\right] = \rho.$$

PROOF. Replacing $f^{(1)}(z)$ and $f(z)$ by $f^{(s)}(z)$ and $f^{(s-1)}(z)$ in (3.1) and taking into consideration that the order of the derivative of an integral function is the same as that of the function, we have

$$\limsup_{r\to\infty}\left[\frac{\log\{r\, M^{(s)}(r)/M^{(s-1)}(r)\}}{\log r}\right] = \rho.$$

From this it easily follows that for $r \geqslant r_0 = r_0(f)$,

$$\frac{M^{(s)}(r)}{M^{(s-1)}(r)} < r^{\rho+\epsilon-1}.$$

Hence

$$r^{(\rho+\epsilon-1)s} > \frac{M^{(s)}(r)}{M^{(s-1)}(r)} \cdot \frac{M^{(s-1)}(r)}{M^{(s-2)}(r)} \cdot \ldots \cdot \frac{M^{(1)}(r)}{M(r)},$$

or

$$r^{(\rho+\epsilon-1)} > \left[\frac{M^{(s)}(r)}{M(r)} \right]^{1/s}.$$

This leads to

$$\limsup_{r\to\infty} \left[\frac{\log r\, \{M^{(s)}(r)/M(r)\}^{1/s}}{\log r} \right] \leqslant \rho. \tag{4.1}$$

Further, Shah [2] has shown that

$$\liminf_{r\to\infty} \left[\frac{\log \{r\, M^{(1)}(r)/M(r)\}}{\log r} \right] = \lambda.$$

Proceeding exactly as above, we get

$$\liminf_{r\to\infty} \left[\frac{\log r\, \{M^{(s)}(r)/M(r)\}^{1/s}}{\log r} \right] \geqslant \lambda = \rho, \tag{4.2}$$

since $f(z)$ is of regular growth. Hence

$$\rho \leqslant \liminf_{r\to\infty} \left[\frac{\log r\, \{M^{(s)}(r)/M(r)\}^{1/s}}{\log r} \right].$$

$$\leqslant \limsup_{r\to\infty} \left[\frac{\log r\, \{M^{(s)}(r)/M(r)\}^{1/s}}{\log r} \right] \leqslant \rho.$$

This proves the theorem.

5. Theorem 4. *If $f(z)$ is an integral function of lower order λ, $\lambda > 1$, then*

$$\limsup_{r\to\infty} \left[\frac{\log r\, \{M^{(s)}(r)/M(r)\}^{1/s}}{\log r} \right] = \rho.$$

Proof. For $\lambda > 1$ and $r \geqslant r_0 = r_0(f)$,

$$M^{(s)}(r) > M(r) \left[\frac{\log M(r)}{\log r} \right]^{s}$$

and hence

$$r \left[\frac{M^{(s)}(r)}{M(r)} \right]^{1/s} > \frac{\log M(r)}{\log r} \tag{5.1}$$

or

$$\limsup_{r\to\infty}\left[\frac{\log r\,\{M^{(s)}(r)/M(r)\}^{1/s}}{\log r}\right] \geqslant \rho. \tag{5.2}$$

Now the result follows from this and the inequality (4.1).

6. Theorem 5. *If $f(z)$ is an integral function of lower order λ, then*

$$\liminf_{r\to\infty}\left[\frac{\log r\,\{M^{(s)}(r)/M(r)\}^{1/s}}{\log r}\right] = \lambda.$$

Proof. Let

$$\theta(r) = \frac{\log r\,\{M^{(s)}(r)/M(r)\}^{1/s}}{\log r}.$$

Proceeding exactly as Shah [2],

$$\liminf \theta(r) \leqslant \lambda.$$

This with the inequality (4.2) gives the result.

7. Theorem 6. *If $f(z)$ is an integral function of order ρ and lower order λ, then, for $r \geqslant r_0 = r_0(f)$,*

$$M(r)r^{(\lambda-\epsilon-1)s} < M^{(s)}(r) < M(r)r^{(\rho+\epsilon-1)s}.$$

This can easily be proved from Theorems 3 and 5.

8. Here we will give an alternative proof of Theorem C.

Writing Vijayaraghavan's [3] inequality as

$$r\,\frac{M^{(s)}(r)}{M^{(s-1)}(r)} \geqslant \frac{\log M^{(s-1)}(r)}{\log r},\ \text{for } r \geqslant r_0,$$

we have

$$r\,\frac{M^{(s)}(r)}{M^{(s-1)}(r)} > \frac{\log M(r)}{\log r}, \tag{8.1}$$

because $M(r) < M^{(s-1)}(r)$, for $\lambda > 1$, from Theorem A, and hence

$$r\left[\frac{M^{(s)}(r)}{M(r)}\right]^{1/s} > \frac{\log M(r)}{\log r} \tag{8.2}$$

or

$$M^{(s)}(r) > M(r)\left[\frac{\log M(r)}{r\log r}\right]^{s},$$

for $r \geqslant r_0$.

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GENERALIZATION OF LIE'S THEOREM ON NULL LINES

By R. N. KAUL and RAM BEHARI

LIE's theorem on null lines in a Euclidean 3-space states that the locus of the middle point of the join of two points, one on each of the two null lines in space, is a minimal surface. It is also known that the locus of a point that divides the join of two points one on each of the two null lines in space, in any constant ratio, is also a minimal surface [1,2].

The object of this note is to generalize this theorem to that in a Euclidean space E_n of n dimensions.

Consider an E_n with orthogonal coordinates $y^\alpha (\alpha, \beta, \ldots = 1, 2 \ldots, n)$ and metric

$$ds^2 = \delta_{\alpha\beta}\, dy^\alpha\, dy^\beta, \tag{1.1}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta. Let

$$y^\alpha = f^\alpha(x^1),\ y^\alpha = g^\alpha(x^2) \tag{1.2}$$

be the equations of two null lines imbedded in E_n. We have then the relations

$$\delta_{\alpha\beta}\, f^\alpha_{,1}\, f^\beta_{,1} = 0,\ \delta_{\alpha\beta}\, g^\alpha_{,2}\, g^\beta_{,2} = 0, \tag{1.3}$$

where the comma denotes partial differentiation with respect to x^i $(i, j, \ldots = 1, 2)$.

The coordinates y^α of the point P which divides the join of any two points f^α and g^α one on each of the two null lines, in a constant ratio $m|n$ are given by

$$y_\alpha = \frac{1}{m+n}(nf^\alpha + mg^\alpha) \tag{1.4}$$

and the metric of this subspace by

$$ds^2 = g_{ij}\, dx^i\, dx^j, \tag{1.5}$$

where

$$g_{ij} = \delta_{\alpha\beta}\, y^{\alpha}_{,i}\, y^{\beta}_{,j} \tag{1.6}$$

and consequently

$$g_{11} = 0,\ g_{12} = \frac{mn}{(m+n)^2}\, \delta_{\alpha\beta}\, f^{\alpha}_{,1}\, g^{\beta}_{,2},\ g_{22} = 0. \tag{1.7}$$

For the Christoffel symbols formed with g_{ij}, we find

$$\left\{ {1 \atop 12} \right\} = 0,\ \left\{ {2 \atop 12} \right\} = 0 \tag{1.8}$$

by virtue of (1.7).

Consequently, for the mean curvature vector, we have [3]

$$\tfrac{1}{2}\, g^{ij}\, y^{\alpha}_{,ij} = \tfrac{1}{2}\, g^{ij} \left(\frac{\partial^2 x^{\alpha}}{\partial x^i\, \partial x^j} - \left\{ {a \atop ij} \right\} y^{\alpha}_{,a} \right) = 0 \tag{1.9}$$

by virtue of (1.2), (1.7) and (1.8).

From (1.9) we get the result that the locus of the points which divide the join of any two points one on each of the two null lines imbedded in a Euclidean E_n in any constant ratio is a minimal variety of 2 dimensions, which is a generalization of Lie's theorem for a surface in a Euclidean 3-space.

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A NOTE ON A FAMILY OF PARALLEL HYPERSURFACES IN A RIEMANNIAN V_n

By M. K. SINGAL and RAM BEHARI

1. Family of parallel hypersurfaces. Let ϕ be a scalar invariant for Riemannian space V_n of n dimensions. Then $\phi =$ const. represents a family of hypersurfaces imbedded in V_n [1]. The unit normal N to the hypersurface at any point is given by

$$N = \psi \nabla \phi \tag{1.1}$$

where

$$\psi^2 = \frac{1}{(\nabla \phi)^2}. \tag{1.2}$$

The necessary and sufficient condition that the hypersurfaces may constitute a parallel system is that rot $N = 0$. But

$$\begin{aligned} \operatorname{rot} N = \operatorname{rot} (\psi \nabla \phi) &= \nabla \phi \times \nabla \psi + \psi \operatorname{rot} (\nabla \phi) \\ &= \nabla \phi \times \nabla \psi. \end{aligned}$$

Therefore the necessary and sufficient condition that the hypersurfaces may constitute a parallel system is that

$$\nabla \phi \times \nabla \psi = 0,$$

i.e. ψ is a function of ϕ only.

2. Mean curvature of a family of parallel hypersurfaces. The mean curvature M of a family of hypersurfaces is given by

$$\begin{aligned} M &= - \operatorname{div} N \\ &= - \operatorname{div} (\psi \nabla \phi) \\ &= - (\psi \nabla^2 \phi + \nabla \phi . \nabla \psi). \end{aligned} \tag{2.1}$$

If the hypersurfaces form a parallel system, ψ is a function of ϕ only, i.e.

$$\psi = f(\phi), \tag{2.2}$$

say. (2.1) then takes the form

$$
\begin{aligned}
M &= -[f(\phi)\nabla^2\phi + f'(\phi)(\nabla\phi)^2] \\
&= -(\nabla\phi)^2[f(\phi)\nabla^2\phi/(\nabla\phi)^2 + f'(\phi)] \\
&= -\{f(\phi)\}^{-2}[\theta f(\phi) + f'(\phi)],
\end{aligned} \tag{2.3}
$$

where

$$\theta = \nabla^2\phi/(\nabla\phi)^2 \tag{2.4}$$

3. Family of isometric (or isothermic) hypersurfaces. A family of hypersurfaces ϕ = constant is said to be isometric if $\theta(\equiv\nabla^2\phi/(\nabla\phi)^2)$ is a function of ϕ only. From (2.3) we find that if a family of parallel hypersurfaces is isometric, M is a function of ϕ only, i.e. the mean curvature is constant over each member of the family.

Hence we have the following

THEOREM. *If a family of parallel hypersurfaces is isometric, the mean curvature is constant over each member of the family.*

For $n = 3$, we have the following known result:

If a family of parallel surfaces is isometric the mean curvature constant over each member of the family [2, page 105].

Again, if M is a function of ϕ only, from (2.3) we find that θ must be a function of ϕ only, i.e. the family must be isometric. Hence we have the following

THEOREM. *If the mean curvature is constant over each member of a family of parallel hypersurfaces, the family is isometric.*

For $n = 3$, we have the following result:

If the mean curvature is constant over each member of a family of parallel surfaces the family is isometric.

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A NOTE ON THE DERIVATIVES OF INTEGRAL FUNCTIONS

By QAZI IBADUR RAHMAN

1. Introduction. Let $f(z) = \sum_{0}^{\infty} a_n z_n$ be an integral function of order ρ and lower order λ, and $\mu(r, f), \mu(r, f'), \ldots, \mu(r, f^{(s)})$ the maximum terms of $f(z), f'(z), \ldots, f^{(s)}(z)$ respectively for $|z| = r$. In a recent paper [2, p.145] Mahendra Kumar Jain has proved the following theorem.

THEOREM. *If $f(z)$ is an integral function of order ρ $(\rho < 1)$ and $\mu(r, f)$ is the maximum term of $f(z)$, then*

$$\mu(r,f) > \mu(r,f') > \mu > (r,f'') > \ldots \mu(r,f^{(s)}) \textit{ for } r > r_0(f). \quad (1.1)$$

The purpose of this note is to prove the following theorem and to deduce a number of interesting results which include (1.1).

THEOREM. *If $\nu(r, f), \nu(r, f'), \ldots, \nu(r, f^{(s)})$ denote the ranks of the maximum terms of $f(z), f'(z), \ldots, f^{(s)}(z)$ respectively for $|z| = r$, then*

$$\nu(r,f) \leqslant r \frac{\mu(r,f')}{\mu(r,f)} \leqslant \nu(r,f'). \quad (1.2)$$

2. Proof of the theorem. If $c_{\nu(r,f)}\, r^{\nu(r,f)}$ and $\nu(r, f')\, c_{\nu(r,f')}\, r^{\nu(r,f')-1}$ are the maximum terms of $f(z)$ and $f'(z)$, then [1, p.35]

$$\mu(r,f') = \nu(r,f')\, c_{\nu(r,f')}\, r^{\nu(r,f')-1} \leqslant \frac{\nu(r,f')}{r}\, c_{\nu(r,f)}\, r^{\nu(r,f)} = \frac{\nu(r,f')}{r}\, \mu(r,f)$$

and the second inequality of (1.2) follows. Also

$$\mu(r,f') = \nu(r,f')\, c_{\nu(r,f')}\, r^{\nu(r,f')-1} \geqslant \nu(r,f)\, c_{\nu(r,f)}\, r^{\nu(r,f)-1} = \frac{\nu(r,f)}{r}\, \mu(r,f)$$

and the first inequality of (1.2) follows.

3. Applications. From (1.2) we can very easily deduce the following results.

$$\nu(r,f) \leqslant \nu(r,f') \leqslant \nu(r,f'') \leqslant \ldots \leqslant \nu(r,f^{(s)}), \quad (3.1)$$

$$\frac{\mu(r,f')}{\mu(r,f)} \leqslant \frac{\mu(r,f'')}{\mu(r,f')} \leqslant \dots \leqslant \frac{\mu(r,f^{(s+1)})}{\mu(r,f^{(s)})}, \tag{3.2}$$

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\left\{ r\dfrac{\mu(r,f')}{\mu(r,f)}\right\}}{\log r} = \begin{matrix}\rho\\ \lambda\end{matrix} = \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log M(r)}{\log r}, \tag{3.3}$$

$$\limsup_{r\to\infty} \frac{\log \nu(r,f)}{\log r} = \rho = 1 + \limsup_{r\to\infty} \frac{\log\left\{\dfrac{\mu(r,f')}{\mu(r,f)}\right\}}{\log r}, \tag{3.4}$$

$$\liminf_{r\to\infty} \frac{\log \nu(r,f)}{\log r} = \lambda = 1 + \liminf_{r\to\infty} \frac{\log\left\{\dfrac{\mu(r,f')}{\mu(r,f)}\right\}}{\log r} \tag{3.5}$$

Let C and C_1 denote two positive constants.

If $\mu(r, f') \geqslant C\,\mu(r, f)$ *for all* $r > r_0$ *then either* $\lambda > 1$ *or* $\lambda = 1$ *and* $\liminf\limits_{r\to\infty} \dfrac{\nu(r, f')}{r} \geqslant C$. (3.6)

From the second inequality of (1.2) we have $\liminf\limits_{r\to\infty} \dfrac{\nu(r, f')}{r} \geqslant C$. Hence $\lambda \geqslant 1$.

If $\mu(r, f') \leqslant C_1\,\mu(r, f)$ *for all* $r > r_0$ *then either* $\rho < 1$ *or*

$$\rho = 1 \textit{ and } \limsup_{r\to\infty} \frac{\nu(r, f)}{r} \leqslant C_1. \tag{3.7}$$

This follows from the first inequality of (1.2).

Let $\phi(r)$ be any function, nondecreasing and positive for $r > r_0$, and such that $\log \phi(r)/\log r$ tends to zero as r tends to infinity.

If $\mu(r, f') \geqslant (1/\phi(r))\,\mu(r, f)$ *for a sequence of values of* r *tending to infinity, then* $\rho \geqslant 1$. (3.8)

This follows from (3.4). If this hypothesis holds for all $r > r_0$ then from (3.5) we get $\lambda \geqslant 1$.

If $\mu(r, f') \leqslant \phi(r)\,\mu(r, f)$ *for a sequence of values of* r *tending to infinity, then* $\lambda \leqslant 1$. (3.9)

This follows from (3.5). If this hypothesis holds for all $r > r_0$ then from (3.4) we get $\rho \leqslant 1$.

If $\frac{1}{\phi(r)} \leqslant \frac{\mu(r, f')}{\mu(r, f)} \leqslant \phi(r)$ *for all* $r > r_0$ *then* $\lambda = \rho = 1$. (3.10)

This follows from (3.4) and (3.5).

If $\rho < 1$ *then*

$$\mu(r, f) > \phi(r)\ \mu(r, f') > \phi^2(r)\ \mu(r, f'') > \dots > \{\phi(r)\}^s\ \mu(r, f^{(s)}) \quad (3.11)$$

for all $r > r_0$.

This follows [see (3.8) above] from (3.4).

If $\lambda > 1$ *then*

$$\mu(r, f) < \frac{1}{\phi(r)}\ \mu(r, f') < \frac{1}{\phi^2(r)}\ \mu(r, f'') < \dots < \frac{1}{\{\phi(r)\}^s}\ \mu(r, f^{(s)}), \quad (3.12)$$

for all $r > r_0$.

This follows from (3.5).

If $\lambda = 1$ *and* $\liminf_{r \to \infty} \frac{\nu(r, f)}{r} > 1$, *then*

$$\mu(r, f) < \mu(r, f') < \mu(r, f'') < \dots < \mu(r, f^{(s)}) \text{ for all } r > r_0. \quad (3.13)$$

If $\liminf_{r \to \infty} \frac{\nu(r, f)}{r} = l > 1$ then, since

$$\frac{\mu(r, f')}{\mu(r, f)} > \frac{\nu(r, f)}{r},$$

$$\liminf_{r \to \infty} \frac{\mu(r, f')}{\mu(r, f)} > 1.$$

Hence for all $r > r_0$

$$\frac{\mu(r, f')}{\mu(r, f)} > 1,$$

$$\frac{\mu(r, f'')}{\mu(r, f')} \geqslant \frac{\mu(r, f')}{\mu(r, f)} > 1,$$

and the complete result follows from (3.2).

If $\lambda = 1$ *and* $\liminf_{r\to\infty} \dfrac{\nu(r, f')}{r} < 1$ *then there is a sequence of numbers r tending to infinity for which*

$$\mu(r, f) > \mu(r, f') > \ldots > \mu(r, f^{(s)}). \tag{3.14}$$

If $\lambda < 1$; *there is a sequence of numbers r tending to infinity for which*

$$\mu(r, f) > \phi(r)\mu(r, f') > \phi^2(r)\mu(r, f'') > \ldots > \{\phi(r)\}^s \mu(r, f^{(s)}). \tag{3.15}$$

If $\rho = 1$ *and* $\limsup_{r\to\infty} \dfrac{\nu(r, f)}{r} > 1$, *there is a sequence of numbers r tending to infinity for which*

$$\mu(r, f) < \mu(r, f') < \mu(r, f'') < \ldots < \mu(r, f^{(s)}). \tag{3.16}$$

If $\rho > 1$, *there is a sequence of numbers r tending to infinity for which*

$$\mu(r, f) < \frac{1}{\phi(r)}\mu(r, f') < \frac{1}{\phi^2(r)}\mu(r, f'') < \ldots < \frac{1}{\{\phi(r)\}^s}\mu(r, f^{(s)}) \tag{3.17}$$

In the end I must thank Dr. S. M. Shah for helpful guidance.

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T. VIJAYARAGHAVAN : ADDENDUM

By K. CHANDRASEKHARAN

THE problem [*Maths. Student*, 24 (1956), 261, 262] posed by R. Salem whether the perfect set P, of Cantor's type, with the constant ratio of dissection $(\xi, 1 - 2\xi, \xi)$, $0 < \xi < \frac{1}{2}$, is a set of uniqueness for trigonometric series if $1/\xi \in S$, where S is the set of algebraic integers θ whose conjugates other than θ have an absolute value less than 1, has since been settled in the affirmative by R. Salem and A. Zygmund, [*Comptes Rendus* (Paris), 240 (1955), 2281-2283].

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MATHEMATICAL NOTES

A note on moments and cumulants

By P. S. DHRUVARAJAN and M. K. SINGAL, *Ramjas College, Delhi*

1. THE following formulae for moments (μ_r) in terms of cumulants (k_r) and for cumulants in terms of moments are well known:

$$\text{(i)} \quad \mu'_r = \sum_{m=0}^{r} \sum \left(\frac{k_{p_1}}{p_1}\right)^{\pi_1} \left(\frac{k_{p_2}}{p_2}\right)^{\pi_2} \cdots \left(\frac{k_{p_m}}{p_m}\right)^{\pi_m} \frac{r\,!}{\pi_1!\,\pi_2!\ldots\pi_m!},$$

where the second summation extends over all non-negative integral values of the π's such that

$$p_1\pi_1 + p_2\pi_2 + \ldots + p_m\pi_m = r;$$

$$\text{(ii)}\ k_r = r\,! \sum_{m=0}^{r} \sum \left(\frac{\mu'_{p_1}}{p_1\,!}\right)^{\pi_1} \left(\frac{\mu'_{p_2}}{p_2\,!}\right)^{\pi_2} \cdots \left(\frac{\mu'_{p_m}}{p_m\,!}\right)^{\pi_m} \frac{(-1)^{\rho-1}(\rho-1)\,!}{\pi_1!\,\pi_2!\ldots\pi_m!},$$

the second summation extending over all non-negative integral values of π's and ρ's subject to

$$p_1\pi_1 + p_2\pi_2 + \ldots + p_m\pi_m = r,$$

and

$$\pi_1 + \ldots + \pi_m = \rho.$$

When we are interested in moments and cumulants of order higher than the first few, the double summation becomes cumbersome. Consequently, in the present note, we have obtained expressions for moments in terms of cumulants and for cumulants in terms of moments in alternative forms which are comparatively easier to work with.

The following generalizations of the expansions of e^x and $\log(1+x)$ given by Mahajani (*J. Univ. Bombay,* 1947) have been used for the purpose:

$$(a)\quad e^{a_1x+a_2x^2+\ldots+a_rx^r+\ldots} = 1 + A_1\,x + A_2\,x^2/2! + \ldots + A_r\,x^r/r! + \ldots,$$

where

$$A_r = \begin{vmatrix} a_1 & -1 & 0 & 0 & \cdot & \cdot & 0 & 0 \\ 2a_2 & a_1 & -2 & 0 & \cdot & \cdot & 0 & 0 \\ 3a_3 & 2a_2 & a_1 & -3 & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ ra_r & (r-1)\,a_{r-1} & (r-2)a_{r-2} & \cdot & \cdot & \cdot & 2a_2 & a_1 \end{vmatrix} \quad (1.1)$$

(b) $$\log(1 + a_1 x + a_2 x^2 + \ldots) = B_1 x - B_2 x^2/2 + B_3 x^3/3 - \ldots + (-1)^{r-1} B_r x^r/r + \ldots,$$

where

$$B_r = \begin{vmatrix} a_1 & 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 2a_2 & a_1 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 3a_3 & a_2 & a_1 & 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ ra_r & a_{r-1} & a_{r-2} & \cdot & \cdot & \cdot & \cdot & a_1 \end{vmatrix} . \quad (1.2)$$

2. Expression for moments in terms of cumulants. If μ'_r denote the rth moment about the origin and k_r denote the rth cumulant, they are connected by the relation,

$$e^{k_1 it + k_2 (it)^2/2! + \ldots + k_r (it)^r/r! + \ldots} = 1 + \mu'_1\, it + \mu'_2 \frac{(it)^2}{2!} + \ldots + \mu'_r \frac{(it)^r}{r!} + \ldots . \quad (2.1)$$

Using (1.1) to expand the left-hand side of (2.1).,

$$1 + A_1 it + A_2 \frac{(it)^2}{2!} + \ldots + A_r \frac{(it)^r}{r!} + \ldots = 1 + \mu'_1 (it) + \mu'_2 \frac{(it)^2}{2!} + \ldots + \mu'_r \frac{(it)^r}{r!} + \ldots, \quad (2.2)$$

where

$$A_r = \begin{vmatrix} k_1 & -1 & 0 & 0 & . & . & . & 0 \\ \frac{k_2}{1!} & k_1 & -2 & 0 & . & . & . & 0 \\ \frac{k_3}{2!} & \frac{k_2}{1!} & k_1 & -3 & . & . & . & 0 \\ . & . & . & . & . & . & . & . \\ \frac{k_r}{(r-1)!} & \frac{k_{r-1}}{(r-2)!} & \frac{k_{r-2}}{(r-3)!} & . & . & . & . & k_1 \end{vmatrix}. \quad (2.3)$$

Now, comparing the coefficients of $(it)^r/r!$ on both sides of (2.2) we have,

$$\mu'_r = A_r. \quad (2.4)$$

(2.4) expresses the rth moment about the origin in terms of cumulants, where A_r is given by (2.3).

The expression for μ'_r in terms of k's may also be obtained in the following way.

The μ's and k's are connected by the relation [M. G. Kendall : *Advanced theory of statistics*, i, pp. 61-63, 67]

$$\mu'_r = \sum_{j=1}^{r} \binom{r-1}{j-1} \mu'_{r-j} k_j. \quad (2.5)$$

Now, (2.5) may be written,

$$\mu'_r = \sum_{j=1}^{r-1} \binom{r-1}{j-1} \mu'_{r-j} k_j + k_r,$$

that is,

$$k_r + \sum_{j=1}^{r-1} \binom{r-1}{j-1} \mu'_{r-j} k_j - \mu'_r = 0. \quad (2.6)$$

Putting $r = 1, 2, 3, \ldots. r$ in (2.6), we get

$$\left.\begin{aligned} & k_1 - \mu'_1 = 0, \\ & k_2 + \binom{1}{1} \mu'_1 k_1 - \mu'_2 = 0, \\ & k_3 + \binom{2}{1} \mu'_1 k_2 + \binom{2}{2} \mu'_2 k_1 - \mu'_3 = 0, \\ & k_r - \mu'_r + \binom{r-1}{1} \mu'_1 k_{r-1} + \binom{r-1}{2} \mu'_2 k_{r-2} + \ldots + \\ & \qquad + \binom{r-1}{r-1} \mu'_{r-1} k_1 = 0. \end{aligned}\right\} \tag{2.7}$$

Eliminating $\mu'_1, \mu'_2, \ldots \mu'_{r-1}$ from equations (2.7) we have

$$\mu'_r = \begin{vmatrix} k_1 & -1 & 0 & 0 & . & . & . & 0 \\ k_2 & \binom{1}{1} k_1 & -1 & 0 & . & . & . & 0 \\ k_3 & \binom{2}{1} k_2 & \binom{2}{2} k_1 & -1 & . & . & . & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ k_r & \binom{r-1}{1} k_{r-1} & \binom{r-1}{1} k_{r-2} & . & . & . & . & \binom{r-1}{r-1} k_1 \end{vmatrix}. \tag{2.8}$$

(2.8) expresses μ'_r in terms of k's in a form slightly different from (2.4). The equivalence of the determinants in (2.3) and (2.8) can be easily established by multiplying the 3rd, 4th, ... rth columns of the determinant in (2.8) by $2!, 3!, \ldots (r-1)!$ respectively and dividing the 3rd, 4th ... rth rows of the same determinant by $2!, 3!, \ldots (r-1)!$ respectively.

3. Expression for cumulants in terms of moments. By taking logarithms on both sides of (2.1), we get

$$k_1(it) + k_2 \frac{(it)^2}{2!} + \ldots + k_r \frac{(it)^r}{r!} + \ldots = \log \left\{ 1 + \mu'_1(it) + \mu'_2 \frac{(it)^2}{2!} + \right. \\ \left. + \ldots + \mu'_r \frac{(it)^r}{r!} + \ldots \right\}. \tag{3.1}$$

Using (b) of §1 to expand the right hand side of (3.1), we have

$$k_1(it) + k_2\frac{(it)^2}{2!} + \ldots + k_r\frac{(it)^r}{r!} + \ldots = B_1(it) - B_2\frac{(it)^2}{2} + $$
$$+ B_3\frac{(it)^3}{3} - \ldots + (-1)^{r-1}B_r\frac{(it)^r}{r} + \ldots, \tag{3.2}$$

where

$$B_r = \begin{vmatrix} \mu'_1 & 1 & 0 & 0 & . & . & . & 0 \\ \frac{\mu'_2}{1!} & \mu'_1 & 1 & 0 & . & . & . & 0 \\ \frac{\mu'_3}{2!} & \frac{\mu'_2}{2!} & \mu'_1 & 1 & . & . & . & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ \frac{\mu'_r}{(r-1)!} & \frac{\mu'_{r-1}}{(r-1)!} & \frac{\mu'_{r-2}}{(r-2)!} & . & . & . & . & \mu'_1 \end{vmatrix}. \tag{3 3}$$

Now from (3.2) we have

$$k_r = (-1)^{r-1}(r-1)!\,B_r. \tag{3.4}$$

(3.4) expresses k_r in terms of the μ's about the origin, where B_r is given by (3.3).

Multiplying the 3rd, 4th ..., rth rows of the determinant on the right hand side of (3.3) by 2!, 3!, ... $(r-1)!$ respectively, and dividing the 4th, 5th, ..., rth columns of the same determinant by 2!, 3!, ... $(r-2)!$ respectively, we get the following known form [Kendall, loc. cit. p. 87] for k_r in terms of μ's :

$$k_r = (-1)^{r-1}\begin{vmatrix} \mu'_1 & 1 & 0 & 0 & . & . & . & 0 \\ \mu'_2 & \binom{1}{0}\mu'_1 & 1 & 0 & . & . & . & 0 \\ \mu'_3 & \binom{2}{0}\mu'_1 & \binom{2}{1}\mu'_2 & 1 & . & . & . & 0 \\ . & . & . & . & . & . & . & . \\ \mu'_r & \binom{r-1}{0}\mu'_{r-1} & \binom{r-1}{1}\mu'_{r-2} & . & . & . & . & \binom{r-1}{r-2}\mu'_1 \end{vmatrix} \tag{3.5}$$

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Acute angle in analytical geometry

By S. S. Subramanyam, *S. V. University College, Tirupathi*

1. **Introduction.** A class of problems occurring in elementary geometry are those referring to the acute (or the obtuse) angle between two given straight lines. The basic principle underlying such problems is not usually discussed in textbooks as it involves the notions of cross-ratio and *separation* of point pairs. I attempt here an approach to such problems.

In Euclidean geometry the straight line is considered open and we can talk of a point C being *between* two given points A and B. In projective geometry, however, the straight line is considered as a closed curve and two points on it will divide it into two segments between which we cannot distinguish. So, here the concept of *betweenness* has to be replaced by that of *seperation* of point-pairs. A real point-pair (A, B) is said to *seperate* another real pair (C, D), if A, B belong to different segments cut out by (C, D). We may also observe that the concept of *betweenness* in Euclidean geometry can be defined in terms of *separation*. Thus a point C is said to lie *between* two point A and B if C and the point at infinity on the line AB seperate (A, B). Otherwise C is said to lie *outside* (AB).

Now, *separation* can be defined algebraically, in terms of cross-ratio, as follows:

DEFINITION. *The pair given by* (x_3, x_4) *is said to separate the pair* (x_1, x_2) *if*

$$(x_1, x_2; x_3, x_4) = \frac{(x_1 - x_3)(x_2 - x_4)}{(x_1 - x_4)(x_2 - x_3)}$$

is negative.

We readily see that the sign of the cross-ratio $(x_1, x_2; x_3, x_4)$ is unaffected by an interchange of (x_1, x_2) or of (x_3, x_4) or of both.

Since *separation* is defined in terms of the sign of the cross-ratio, it follows that separation is a symmetric relation between two *unordered* point-pairs. Also since cross-ratio is invariant terms fractional linear as well as for projective transformations, the relation of separation is a projective invariant.

2. The following properties of seperation follow as a consequence of the definition.

THEOREM 1. *Four distinct points A, B, C, D on a straight line can be partitioned into two pairs in three ways; and in one and only one of these cases do we get two separating pairs.*

The three partitions are $(AB; CD)$, $(AC; BD)$ and $(AD; BC)$. Now, if

$$(AC; BD) = (ABCD) = \lambda,$$

then

$$(AB; CD) = (ACBD) = 1 - \lambda,$$

and

$$(AD; BC) = (ABDC) = \lambda/(\lambda - 1).$$

If λ is positive the other two are of opposite signs; if λ is negative the other two are both positive. Thus *one and only one* of the three cross-ratios is negative, which proves our statement.

COROLLARY. *In particular, if the pair* (A, C) *is harmonic to the pair* (B, D), *then* (A, C) *separates* (B, D). *Therefore the other two partitions* $(AB; CD)$ *and* $(AD; BC)$ *give non-separating pairs.*

THEOREM 2. *If a point-pair (A, B) separates (or does not separate) real point-pair (C, D), then their common harmonic pair is complex (or real); also any two pairs belonging to the involution determined by (A, B) and (C, D), separate or do not separate each other according as (A, B) and (C, D) separate or do not separate each other.*

LEMMA. *If (x_1, x_2) is a real point-pair, the pair (x, ∞) separates or does not separate (x_1, x_2) according as*

$$(x_1, x_2; x, \infty) \lessgtr 0,$$

that is, as

$$\frac{(x_1 - x)(x_2 - \infty)}{(x_1 - \infty)(x_2 - x)} \lessgtr 0 \text{ or, as } (x - x_1)(x - x_2) \lessgtr 0. \tag{2.1}$$

These are respectively the conditions for x to lie "between" and "outside" $(x_1, x_2.)$

On account of the invariance of the relation of separation we can take, without loss of generality, the two real point-pairs as given by

$$(A, B) \equiv x^2 - p^2 = 0, (C, \infty) \equiv 2bx + c = 0.$$

By the Lemma, these two pairs will or will not separate each other according as

$$(-c/2b - p)(-c/2b + p) \lessgtr 0,$$

that is, as

$$4b^2 p^2 - c^2 \lessgtr 0. \tag{2.2}$$

Now, c is the *apolar invariant* of the two quadratics, $-2p^2$ and $-2b^2$ are the *discriminants* of the two quadratics respectively (see Note 1 below).

[NOTE 1. Symbolically, let two quadratics be defined by

$$\alpha_x^2 \equiv a_0x^2 + 2a_1x + a_2, \text{ and } \beta_x^2 \equiv b_0x^2 + 2b_1x + b_2.$$

Their *apolar invariant* is defined as

$$H \equiv (\alpha\beta)^2 = a_0b_2 + a_2b_0 - 2a_1b_1.$$

When the two quadratics are the same quadratic f, the apolar invariant is called the *discriminant* of f. Using equivalent symbols, α', β', the discriminants of α_x^2 and β_x^2 are given by

$$D \equiv (\alpha\alpha')^2 = 2(a_0 a_2 - a_1^2), \text{ and } D' \equiv (\beta\beta')^2 = 2(b_0 b_2 - b_1^2).$$

Hence if the *discriminant* of a quadratic is negative the quadratic has real roots and if it is positive, the quadratic has complex roots.]

The common harmonic pair (X, Y) of (A, B) and (C, D) is given by the Jacobian (see Note 2 below) of the two quadratics, that is, by

$$bx^2 + cx + bp^2 = 0,$$

and we find that $\frac{1}{2}(DD' - H^2)$ is the discriminant (see Note 2 below) of the Jacobian where H is the apolar invariant, D and D' are the discriminants of the two quadratics; but the conditions (2.2) can be expressed in an invariant form as $(DD' - H^2) \lesseqgtr 0$; and hence the conditions (2.2) are those for the common harmonic pair to be complex or real (see Note 1) hence the first part of the theorem.

[NOTE 2. The Jacobian of two quadratics α_x^2, β_x^2 is given by

$$(\alpha\beta)\,\alpha_x\beta_x \equiv (a_0 b_1 - a_1 b_0)\,x^2 + (a_0 b_2 - a_2 b_0)\,x + (a_1 b_2 - a_2 b_1),$$

and, when equated to zero, represents the common harmonic pair of α_x^2, β_x^2. Now, using equivalent symbols α', β', the discriminant of the Jacobian is given by

$$\{(\alpha\beta)\,\alpha_x\beta_x, (\alpha'\beta')\,\alpha'_x\beta'_x\}^2 \equiv (\alpha\beta)(\alpha'\beta').\tfrac{1}{2}\{(\alpha\alpha')(\beta\beta') + (\alpha\beta')(\beta\alpha')\}$$
$$= \tfrac{1}{2}\{(\alpha\beta)(\alpha'\beta')\{2(\alpha\alpha')(\beta\beta') - (\alpha\beta)(\alpha'\beta'\},$$

since $\qquad (\alpha\beta')(\beta\alpha') \equiv (\alpha\alpha')(\beta\beta') + (\alpha\beta)(\beta'\alpha'),$

$$= \tfrac{1}{2}\{2(\alpha\beta)(\alpha'\beta')(\alpha\alpha')(\beta\beta') - H^2\}.$$

Let $(\alpha\beta)\ (\alpha'\beta')\ (\alpha\alpha')\ (\beta\beta') = k$. Then

$$k = (\alpha\alpha')(\beta\beta')\{\alpha\alpha')(\beta\beta') - (\alpha\beta')(\beta\alpha')\},$$

$$\text{since } (\alpha\beta)(\alpha'\beta') \equiv (\alpha\alpha')(\beta\beta') - (\alpha\beta')(\beta\alpha'),$$

$$= (\alpha\alpha')^2(\beta\beta')^2 - (\alpha\alpha')(\beta\beta')(\alpha\beta')(\beta\alpha'), \text{ interchanging } \alpha \text{ and } \alpha'$$

$$= DD' - (\alpha\alpha')(\beta\beta')(\alpha'\beta')(\alpha\beta) = DD' - k,$$

so that $k = DD'/2$. Hence the discriminant of the Jacobian $= \frac{1}{2}(DD' - H^2)$, where D, D' are the discriminants of α_x^2, β_x^2 and H is their apolar invariant.]

Conversely, if the common harmonic pair of two real point-pairs be complex (or real), then the two point-pairs separate (or do not separate) each other. Now, consider the involution determined by (A,B), (C,D); the common harmonic pair (X,Y) is the same for any two pairs belonging to the involution. Hence any two pairs of the involution separate or do not separate each other according as (X,Y) is complex or real, that is, according as (A,B) and (C,D) separate or do not separate each other.

THEOREM 3. *If (A,B) is a real point-pair and A_1, A_2, A_3 are three more points collinear with A, B, then out of the point-pairs (A_1, A_2), (A_2, A_3) and (A_3, A_1) either exactly two separate (A,B) or none at all.*

For,

$$(AB;\, A_1A_2) = \frac{AA_1 . BA_2}{AA_2 . BA_1},$$

$$(AB;\, A_2A_3) = \frac{AA_2 . BA_3}{AA_3 . BA_2},$$

and

$$(AB;\, A_3A_1) = \frac{AA_3 . BA_1}{AA_1 . BA_3},$$

the product of these three cross-ratios is unity. Hence all the three must be positive or two negative and one positive, proving our statement.

We can generalize the above result and say that if (A,B) be a real point-pair and A_1, A_2, $A_3, \ldots,$ A_n are n more points on AB then out of the pairs (A_1, A_2), $(A_2, A_3), \ldots,$ (A_{n-1}, A_n) and (A_n, A_1) exactly an even number, or none at all, separate (A,B).

COROLLARY. *It follows from the theorem that if (A,B) and (A_1, A_2) are two non-separating pairs of points on a line, and if A_3 is a point on the line such that (A_3, A_1) separates (or does not separate) (A,B) then (A_3, A_2) also separates (or does not separate) (A,B).*

3. The totality of lines through a point constitute a projective domain; they are defined by a single parameter tan θ, where θ is the angle made by any line with a fixed direction. The concept of *separation* applies equally well for the lines also.

Now, any two lines OA, OB through a point O and their perpendiculars OC, OD belong to an orthogonal involution whose double lines are the circular lines through O. Hence, by Theorem 2 OA, OC separate OB, OD. By the corollary of Theorem 1, OC, OD do not separate OA, OB.

DEFINITION. *The acute (or the obtuse) angle between* OA, OB *is defined as the totality of lines which are such that any of them, say* OP, *and one of the perpendiculars, say* OC, *separate (or do not separate)* OA, OB; *then, by Theorem* 3, *Corollary,* OD *will also separate (or will not separate,* OA, OB.

Now we consider two problems.

PROBLEM 1. To find the conditions that a given line $y = mx$ may lie in the acute or the obtuse angle between two other given lines $y = m_1x$ and $y = m_2x$.

From the definition of the acute angle the line OP $(y = mx)$ lies in the acute or the obtuse angle between OA, $(y = m_1x)$ and OB, $(y = m_2 x)$, according as

$$(m_1,\ m_2;\ m,\ -1/m_1) \lesseqgtr 0,$$

that is, as

$$(m - m_1)\ (m - m_2)\ (1 + m_1 m_2) \lesseqgtr 0. \qquad (3.1)$$

In particular, the y-axis OY lies in the acute or the obtuse angle according as $1 + m_1m_2 \lesseqgtr 0$. (3.2)

Evidently the conditions (3.1) depend on the signs of $(m - m_1) \times$ $\times$ $(m - m_2)$ and $(1 + m_1m_2)$, that is, on whether m lies *between* or *outside* m_1, m_2 and whether the y-axis lies in the acute or the obtuse angle.

The conditions can be obtained alternatively as follows: consider the three lines OP, OY and the perpendicular, say OC, to one of the lines. Out of these lines we must have, by Theorem 3, either exactly two pairs separating OA, OB or none at all. Consider the two pairs OP, OY and OY, OC. When both these pairs separate or both do not separate OA, OB (in this case $(m - m_1)(m - m_2)$ and $(1 + m_1 m_2)$ are of the same sign) the third pair OP, OC does not separate OA, OB; hence, by definition, OP lies in the obtuse angle. When only one of the two pairs separates OA, OB (in this case $(m - m_1)(m - m_2)$ and $(1 + m_1 m_2)$ are of opposite signs), the third pair OP, OC must also separate OA, OB, so that by definition, OP lies in the acute angle.

The results can be tabulated as follows.

Case	m lies	Sign of $1 + m_1 m_2$	$y = mx$ lies in the
1	between m_1, m_2	$+$	acute angle
2	between m_1, m_2	$-$	obtuse angle
3	outside m_1, m_2	$+$	obtuse angle
4	outside m_1, m	$-$	acute angle

(3.3)

PROBLEM 2. To find which of the bisectors of the angles between the lines $y = m_1 x$ and $y = m_2 x$ lies in the acute angle between them.

The two bisectors separate OA, OB because they are harmonic conjugates with respect to OA, OB. The equations to the two bisectors are

$$(B_+): \ \frac{y - m_1 x}{\sqrt{(1 + m_1^2)}} = + \frac{y - m_2 x}{\sqrt{(1 + m_2)}}$$

and

$$(B_-): \frac{y - m_1 x}{\sqrt{(1 + m_1^2)}} = - \frac{y - m_2 x}{\sqrt{(1 + m_2^2)}},$$

where $\sqrt{(1 + m_1^2)}$, and $\sqrt{(1 + m_2^2)}$ are taken both positive.

These equations can be written as $y = mx$ and $y = m'x$ respectively, where

$$m = \frac{m_1 \sqrt{(1 + m_2^2)} - m_2 \sqrt{(1 + m_1^2)}}{\sqrt{(1 + m_2^2)} - \sqrt{(1 + m_1^2)}}$$

and

$$m^1 = \frac{m_1 \sqrt{(1 + m_2^2)} + m_2 \sqrt{(1 + m_1^2)}}{\sqrt{(1 + m_2^2)} + \sqrt{(1 + m_1^2)}}$$

Since $\sqrt{(1 + m_1^2)}$ and $\sqrt{(1 + m_2^2)}$ are both positive, m lies *outside* m_1, m_2 and m lies *between* m_1, m_2. Hence, using Table (3.3), we see that the bisector $y = mx$, that is (B_+), lies in the acute angle if $1 + m_1 m_2 < 0$; the other bisector lies in the obtuse angle. And when $1 + m_1 m_2 < 0$ the y-axis lies in the acute angle. Summing up, we have

Sign of $1 + m_1 m_2$	y-axis lies in the	Bisector (B_+) lies in the	Bisector (B_-) lies in the
$1 + m_1 m_2 > 0$	obtuse angle	obtuse angle	acute angle
$1 + m_1 m_2 < 0$	acute angle	acute angle	obtuse angle

(3.4)

I am deeply indebted to Dr. R. Vaidyanathaswamy for his valuable suggestions and criticism throughout the preparation of this note.

On some theorems on congruence

By M. BHASKARAN, *Loyola College, Madras*

1. P. KESAVA MENON has proved the following theorem [*J. Indian Math. Soc.* Vol 3, 1937].

Let p be an odd prime. Then

$$1^{p-1} + 2^{p-1} \ldots + (p-1)^{p-1} - p - (p-1)! \equiv 0 \pmod{p^2}.$$

In this note, I am giving a similar and more general method for proving this theorem.

PROOF. Let

$$(x-1)(x-2)\ldots(x-(p-1)) \equiv x^{p-1} + A_1 x^{p-2} + \ldots + A_{p-1},$$

where p is prime. Let S_n denote the sum of the nth powers of the roots of $x_{p-1} + \ldots + A_{p-1} = 0$. Then by Newton's theorem,

$$S_1 + A_1 = 0;$$

$$S_2 + A_1 S_1 + 2A_2 = 0;$$

....

$$S_{p+1} + A_1 S_{p-2} + \ldots + A_{p-2} S_1 + (p-1) A_{p-1} = 0. \qquad \text{(A)}$$

By Lagrange's theorem, we have $A_1, A_2, \ldots A_{p-2} \equiv 0 \pmod{p}$ and

$$A_{p-1} = (p-1)! \equiv -1 \pmod{p}. \qquad \text{(B)}$$

Using this, we easily get

$$S_1, S_2, \ldots, S_{p-2} \equiv 0 \pmod{p}. \qquad \text{(C)}$$

From (A), (B) and (C), it follows that

$$S_{p-1} + (p-1) A_{p-1} \equiv 0 \pmod{p^2},$$

i.e. $$S_{p-1} - (p-1)! - p \pmod{p^2} \equiv 0 \pmod{p^2},$$

or $$S_{p-1} - p - (p-1)! \equiv 0 \pmod{p^2}.$$

2. We give now a generalization of Lagrange's theorem.

Let

$$(x+a)(x+a+1)\ldots(x+a+n-1) \equiv x^n + A_1 x^{n-1} + \ldots + A_n,$$

a any integer. Then all A_i *except* $A_{(p-1)}, A_{2(p-1)}, \dots, A_{(p-1)n/p}$ *are congruent to* 0 (mod p) *where* p *is prime and* $p|n$. *Also*

$$1 + A_{p-1} + \dots + A_{(n/p)(p-1)} \equiv 0 \pmod p.$$

PROOF. Since $a, a+1, \dots, a+(n-1)$ form a complete system of residues modulo n (a being any integer), this system can be split up into n/p groups, each group containing p integers such that each is a complete system of residues modulo p. Let the members of one such group be $\alpha_1, \alpha_2, \dots \alpha_p$. One among the α's is congruent to 0 (mod p). The other α's are roots of the polynomial $(x^{p-1}-1)$ mod p (Fermat's Theorem). Hence the members of the group are roots of the polynomial $(x^p - x)$ mod p. Similarly for other groups. It immediately follows that

$$\begin{aligned} x^n + A_1 x^{n-1} + \dots + A_n &\equiv (x+a)\dots(x+\overline{a+n-1}) \\ &\equiv (x^p - x)^{n/p} \pmod p. \end{aligned}$$

Expanding $(x^p - x)^{n/p}$ it is seen that all A_i except $A_{(p-1)}, \dots A_{2(p-1)}\ A_{(p-1)n/p}$ are congruent to 0 (mod p). Putting $x = 1$ in the above identity it is immediate that

$$1 + A_{p-1} + \dots + A_{n/p(p-1)} \equiv 0 \pmod p.$$

3. $a^k + (a+1)^k + \dots + (a+n-1)^k \equiv 0 \pmod p$ *if* $k \not\equiv 0 \pmod{p-1}$; *here* a, k *and* n *positive integers,* p *prime and* $p|n$.

PROOF. Let σ_i denote the sum of the ith powers of the roots of the polynomial $(x^{p-1}-1)$ mod p;

$$\sigma_i \equiv 1^i + 2^i + \dots + (p-1)^i \pmod p.$$

Evidently $\sigma_i \equiv \sigma_j \pmod p$, $i \equiv j \pmod{p-1}$. This means that

$$\begin{aligned} a^{\lambda(p-1)+\mu} + (a+1)^{\lambda(p-1)+\mu} + \dots + (\overline{a+n-1})^{\lambda(p-1)+\mu} \\ \equiv (n/p)\, \sigma_\mu \pmod p, \end{aligned}$$

($\lambda \geqslant 0$ integer and $1 \leqslant \mu \leqslant p-1$) and this is congruent to 0 (mod p) if $\mu \neq p-1$ since $\sigma_\mu \equiv 0 \pmod p$.

From this, the result follows.

CLASSROOM NOTE

A note on invariants

By K. A. Viswanathan, *Karnatak College, Dharwar*

The following is an alternative proof of the well-known result in Analytical geometry [*vide* E. H. Askwith: *Analytical geometry*, p. 369].

If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ *where* x *and* y *refer to some cartesian axes inclined at an angle* ω *be transferred to* $a'X^2 + 2h'XY + b'Y^2 + 2g'X + 2f'Y + c'$ *where* X *and* Y *refer to some cartesian axes inclined at an angle* ω' *with the same or different origin, then* $\Delta/\sin^2\omega = \Delta'/\sin^2\omega'$ *where*

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}, \quad \Delta' = \begin{vmatrix} a' & h' & g' \\ h' & b' & f' \\ g' & f' & c' \end{vmatrix}$$

The discussion of the invariant $\Delta/\sin^2\omega$ has not to be postponed until homogeneous coordinates are introduced, as in Askwith. It is best done as shown here and can be taken up along with the twins $(a + b - 2h\cos\omega)/\sin^2\omega$ and $(ab - h^2)/\sin^2\omega$.

Let the equations of transformation be

$$x = kX + lY + m, \quad y = k'X + l'Y + m'.$$

Since the expression $ax^2 + 2hyx + by^2 + 2xg + 2fy + c$ transforms to $a'X^2 + 2h'XY + b'Y^2 + 2g'X + 2f'Y + c'$, the expression $ax^2 + 2hxy + by^2 + 2gx + 2fy + c - \lambda$, where λ is a constant, will transform to $a'X^2 + 2h'XY + b'Y^2 + 2g'X + 2f'y + c' - \lambda$. If λ be chosen so that the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c - \lambda = 0$ represents a pair of straight lines then the equation $a'X^2 + 2h'XY + b'Y^2 + 2g'X + 2f'Y + c' - \lambda = 0$ will represent a pair of straight lines.

Since $(ab - h^2)\lambda - \Delta = 0$ and $(a'b' - h'^2)\lambda - \Delta' = 0$ are satisfied for the same value of λ, it is easily seen that we may suppose $ab - h^2 \neq 0$, $a'b' - h'^2 \neq 0$. And clearly the case $\Delta = 0$ or $\Delta' = 0$ is trivial.

This gives us the relation

$$\lambda = \Delta/(ab - h^2) = \Delta'/(a'b' - h'^2).$$

Hence we derive the relation

$$\frac{\Delta}{\Delta'} = \frac{ab - h^2}{a'b' - h'^2} = \frac{\sin^2 \omega}{\sin^2 \omega'},$$

which gives the required result.

BOOK REVIEWS

Differential Calculus. By W. L. Ferrar, Oxford (Clarendon Press), pp. xx + 296, 27sh. 6d.

THIS book is intended as a text-book for first-year graduate students at a university. The material included is that part of mathematical analysis prescribed for honours students of Indian universities with the omission of the theory of Riemann integration and certain theorems on infinite series and uniform convergence. The book is very clearly written and from the point of view of an average student, this book will be helpful. But no attempt has been made to incorporate in the book some of the modern ideas, for example, from point-set theory, suited to the level of undergraduate students, so that the book has not anything to offer apart from what is found in the classical text-books on analysis. Notions of uniform convergence etc. do not belong to differential calculus proper in the opinion of the author, and there is frequent reference to his book on Convergence, as for example, in the chapter on particular functions. In one of the final chapters the author tries to make the notion of second and higher differentials of a function of several variables clear, but not sufficiently enough to illustrate either the necessity or the fruitfulness of this concept, and in fact the reviewer cannot help feeling why this concept should be introduced at all.

C. S. SESHADRI

An introduction to analytical conics. By C. T. Rajagopal and V. R. Srinivasaraghavan, Oxford University Press, Madras, 1955, pp. xxi + 320, Rs. 5-8.

THIS book is intended for the B.A. and B.Sc. classes of Indian universities. No apology is needed for this book since this is of a

type different from the usual books following the 'well-worn pattern'. This has a stamp of its own, since it breaks away from the traditional method both in its treatment and notation. At an early stage, the abridged notation for points and lines is introduced and also the circular points and lines in the complex plane. The idea of an envelope as a natural result of the principle of duality is built up gradually. A special feature is the systematic use of abridged notation 'treated not only as a kind of short hand, but also as a tidy tool for handling many types of geometric configurations'. Another feature which adds to the value of the book is the introduction of the historic development of the subject from the earliest to modern times in suitable stages found throughout.

There are nine chapters and nine appendices. The first chapter introduces the abridged notation so effectively used by H. F. Baker and others, and its elegant application to harmonic ranges and pencils and quadrilaterals. Invariants arising out of the transformation of the axes are also proved. The second chapter takes the student to the principle of duality and tangential equations. Complex geometry is discussed at some length and the theorem that a second degree equation through the circular points at infinity is a circle is proved. The third chapter on circles introduces the notation used for finding the equation to the chord, tangent, polar etc. of a circle. After a study of a pencil of circles and a net of circles we have proofs of Hesse's theorem and Feurbach's theorem.

Chapter IV gives the definition of the conic and the derivation of the standard forms of the three types by the ordinate method, viz. $PN^2 \propto MP.\, M'P$ in preference to the orthodox focus-directrix method. The two-focus property $SP \pm S'P$ is constant is also discussed. The relative merits of the three modes of defining the conic are considered and the ordinate method is preferred since the conic comes naturally as a transition from the circle. The reduction of the general equation to the standard form and the tracing of the conic is treated in great detail with a number of examples.

The next chapter on properties of the general conic gives full play to the notation built up earlier in the case of circles. The notation is this. If the conic is $S = 0$, the tangent at (x_1, y_1) on it is $S_1 = 0$, $S_{12} = 0$ if (x_2, y_2) is on $S_1 = 0$, $S_{11} = 0$ if (x_1, y_1) is on $S = 0$, etc. With the help of this, equations to the chord, tangent, polar of a point, chord of contact, pair of tangents are deduced in a logical way. There is a fear that amidst the mass of symbols the beginner may lose sight of the principles underlying the result. Joachimstal's ratio method is also given. But the well-known method of r-equation is not mentioned at all.

Various properties of the parabola, ellipse and hyperbola are discussed at length in the next three chapters. The last chapter gives a detailed treatment of the conic in polar coordinates. The appendices contain among other things chapters on determinants and on the abridged notation for the conic. Many interesting results are deduced by this method and among them Bobillier's proof of Pascal's theorem. Contact of conics and confocal conics which are treated fully might well have been placed in the body of the book.

The book contains a large number of worked examples. Profuse hints are given for others which cover a wide range. There is such a wealth of material in these that the whole field of geometry is covered—four-point conics, reciprocation, principle of inversive geometry, cross-ratio property, etc. In some of the worked examples, the method of pure geometry as an alternative may have been suggested. For instance the example on p. 204 comes out by argument in a single step. A unique feature is that there are a number of references to several problems and discussions in various journals creating an interest in the pupils and broadening their outlook.

The book is well got up and although the examples are in smaller type there are few misprints. In addition to those pointed out by the authors, here are a few : p. 69, 5 up for N read I, p. 65 (26) for $2\mu\nu$ read $2\mu y$, p. 110 for $\overline{\Delta}|\sin^2 \omega$ read $\Delta|\sin^2 \omega$, p. 128, line 20 should

be $\Sigma\lambda_1$ (2, 3) $= 0$, p. 181 (iii) for $4\,f\,a\,t$ read $4\,f$. A good index will have enhanced the value of the book. We heartily recommend this book to all colleges and libraries.

S. MAHADEVAN

Astrolabes. By M. P. Khareghat, M. P. Khareghat Memorial Volume II, Edited by D. Kapadia, Bombay, 1950.

[Papers containing a description of an astrolabe given to the late M. P. Khareghat by J. B. Petit, edited by D. D. Kapadia.]

REPORT OF THE TWENTY-SECOND CONFERENCE OF THE INDIAN MATHEMATICAL SOCIETY

THE INDIAN MATHEMATICAL SOCIETY

TWENTY-SECOND CONFERENCE, BARODA 1956

PROGRAMME

Monday, December 24, 1956

10-00 A. M.—Inauguration meeting

12-30 P. M.—Lunch

2-15 P. M.—Business meeting of the Council of the Indian Mathematical Society

2-30 P. M.—Annual general meeting of the Indian Mathematical Society

2-45 P. M.—Invited address by Professor K. Yosida, University of Tokyo, on 'Reflexivity in the space of distributions'

3-45 P. M.—Popular lecture by Dr. R. Narasimhan, Tata Institute of Fundamental Research, on 'Computing machines'

5-00 P. M.—At Home by the Reception Committee on the Faculty of Science Lawns

6-30 P. M.—Entertainment programme at Music College, Sursagar

8-00 P. M.—Dinner

Tuesday, December 25, 1956

9-00 A. M.—Reading of papers

10-30 A. M.—Invited address by Dr. M. Venkataraman, University of Madras, on 'Spectral theory'

11-30 A. M.—Reading of papers

1-00 P. M.—Lunch

3-00 P. M.—Invited address by Dr. A. V. Prasad, Ranchi, on 'Geometry of numbers'

4-00 P. M.—Tea for delegates in the Students' common room

5-30 P. M.—Invited address by Dr. U. N. Singh, Muslim University, Aligarh, on 'Generalized Fourier transform and its applications'

6-00 P. M.—Visit to All-India Radio Station

7-30 P. M.—Entertainment programme at Music College

8-30 P. M.—Dinner

Wednesday, December 26, 1956

9-00 A. M.—Reading of papers

11-30 A. M.—Symposium on 'Differential geometry of Riemannian and complex manifolds'

1-00 P. M.—Lunch

2-30 P. M.—Visit to Alembic Chemical Works

5-30 P. M.—Popular lecture by Professor H. Gupta, Panjab University, on 'The Indian Calendar'

8-00 P. M.—Dinner

REPORT OF THE CONFERENCE

Inauguration

The Twenty-second Conference of the Indian Mathematical Society met at Baroda on December 24-26, 1956 under the auspices of the Maharaja Sayajirao University of Baroda. Shrimati Hansa Mehta, Vice-Chancellor of the University, inaugurated the Conference and Professor Ram Behari, President of the Indian Mathematical Society, presided over all the sessions, which were attended by well over a hundred delegates.

Secretary's Annual Report

Professor S. Mahadevan, Secretary of the Indian Mathematical Society, presented his annual report. He conveyed the Society's gratitude to the M. S. University of Baroda for their kind invitation to hold the Twenty-second Conference at Baroda, and for the excellent arrangements they had made.

The Secretary announced that during the year under report the Society lost three of its esteemed members: Professor H. F. Baker, Shri V. Krishnamurti Rao and Professor E. T. Whittakar, and placed on record the Society's sense of deep sorrow at the loss.

Proceeding to the working of the Society, the Secretary mentioned that both the periodicals published by the Society continued to be under the able editorship of Professor K. Chandrasekharan, and congratulated the Commercial Printing Press, Bombay, on the excellent care and vigilance bestowed by them on the printing of these periodicals.

Regarding the Society's Library the Secretary remarked, "We have now about 600 books and 150 periodicals from different countries and about 2000 bound volumes of back numbers of periodicals. The Library is located at the Ramanujan Institute of Mathematics and is very well managed by Professor C. T. Rajagopal, Director of the Institute. It is widely used by the members and research students and I appeal once again to members outside Madras to make use of

the Library to the fullest extent and also to get important periodicals by circulation. We spend annually about Rs. 3,500 towards the improvement of the Library."

The Secretary next referred to the recent International Colloquium on Zeta-functions and the South Asian Conference on Mathematical Education. "Both these were organized in February last by the Tata Institute of Fundamental Research with the help of the International Mathematical Union, UNESCO, the Government of India, and the Sir Dorabji Tata Trust. ... Our Society had the privilege of publishing the reports in full—the proceedings of the Colloquium on Zeta-functions in the *Journal* and those of the South Asian Conference in the *Student*.... In this connection I am glad to state that the Prime Minister in his message to the above Conferences announced that the Government of India have recognized the Tata Institute of Fundamental Research as the national centre for advanced study and fundamental research in mathematics." The Secretary, however, expressed the opinion that for a big country like ours, one such national institute was not enough and mentioned that the Ramanujan Institute of Mathematics in Madras was languishing for want of support to carry out its many projects.

The Secretary continued, "Reagarding our finances, we manage mostly by the voluntary dues from members. I wish to thank the Universities of Bombay and Madras, Osmania University, Tata Institute of Fundamental Research and the Government of India for their annual grants."

The Secretary was grateful to Professor K. Yosida of the University of Tokyo for having kindly agreed to attend the Conference and deliver an address.

Award of the Narasingarao Medal

The Narasingarao Medal for the year 1955 was presented to Mr. T. P. Srinivasan for his paper on "Extension of measures" published in the *Journal of the Indian Mathematical Society*, 19, 1955.

ADDRESS

At the invitation of the President, Professor V. V. Narlikar delivered an address which is printed separately.

MEETING OF THE SOCIETY

The Council and the General Body of the Indian Mathematical Society met in the afternoon of December 24, 1956.

PROCEEDINGS OF THE CONFERENCE

The mathematical programme consisted of presentation of papers, invited addresses and a symposium. There were three sessions devoted to the reading of papers at which about sixty papers were presented. Abstracts of these appear elsewhere. Among the invited addresses were one by Professor K. Yosida on 'Reflexivity in the space of distributions', one by Dr. M. Venkataraman on 'Spectral theory', one by Dr. A. V. Prasad on 'Geometry of numbers' and one by Dr. U. N. Singh on 'Generalized Fourier transform and its applications'. A symposium on 'Differential geometry of Riemannian and complex manifolds' was organized on December 26 in which Professor Ram Behari, Mr. R. N. Kaul and Miss Gita Haldar, among others, participated. A popular lecture on 'Computing machines' was delivered by Dr. R. Narasimhan on December 24 and one on 'the Indian Calendar' by Professor H. Gupta on December 26. As an adjunct to the Conference a mathematical exhibition was got up; the exhibits included charts and drawings of geometrical patterns prepared by the pupils of the District Board High School, Periyakulam.

SOCIAL PROGRAMME

The delegates were lavishly entertained at receptions and musical programmes. They spent a particularly enjoyable evening on December 25 as guests of the All-India Radio, Baroda.

FROM THE TRIVIAL TO THE NON-TRIVIAL THROUGH ISOMORPHISMS BETWEEN THOUGHT AND NATURE

By PROFESSOR V. V. NARLIKAR

"The study of nature, which was formerly one of the main sources of great mathematical problems, seems in recent years to have borrowed from us more than it has given us." *André Weil**

THIS is a great occasion bringing together mathematicians from all parts of our country and the task that I have been invited to perform means both an honour and a responsibility. When the invitation came I was taken by surprise, and conscious of my limitations, I wondered like Nachiketas, what special purpose I was called upon to serve. It was comforting to recall what is supposed to have happened on the day of Solomon's wedding. There was a long procession of the wedding guests who had come from many parts of the earth. There, in a place of honour, was an ant that had also brought a gift, a leaf of grass. Judged by ordinary standards what can be more trivial than a leaf of grass ? And what else, but leaves of mathematical grass, would you expect from a student and a teacher of the 'trivial' in mathematics ?

Yet, it is a fact that under the magic mathematical touch of great masters the trivial has often become important : it has provided novel isomorphisms between thought and Nature, fresh clues for new mathematical developments and some challenging problems which fall beyond the ambit of known mathematics.

Even for those who are not interested in unravelling the secrets of Nature and who wish to pursue the most harmless and 'useless' of mathematics for its own sake, there are unfailing sources of provocation and inspiration in the unsolved problems of applied

* The Future of Mathematics, *American Math. Monthly*, 1950, 57, 295.

mathematics. We have in them a *motivation* for new mathematical discoveries and creations.

I welcome therefore this opportunity to present a few thoughts on the theme, summed up in the phrase, 'from the trivial to the non-trivial'. I would also refer to some unsolved problems in my line of work which deserve the attention of mathematicians.

In these days of thinking more and more, about less and less, the answers that we receive from the specialists even to simple questions are highly complicated. It seems that to know anything one has to know everything. Authorities differ as to what is simple and what is not simple. Contrary to the popular supposition of the logical simplicity of Euclidean geometry it can be shown that, in a certain sense, the hyperbolic geometry initiated by Bolyai and Lobatschevsky is simpler still. In answering the question, "What is geometry ?", Veblen and Whitehead (1932) say : "a branch of mathematics is called a geometry because the name seems good on emotional and traditional grounds, to a sufficient number of competent people". Similarly, the answers to questions like (1) "What is mathematics ?" (2) "What is light ?" and (3) "What is a fundamental particle ?" are far from simple. The study of all these questions has, however, one common feature. We are concerned with structures in each case.

In mathematics we are concerned with logical structures of hypothetical propositions which are said to be true when they are deducible from a set of basic propositions. This must be distinguished from science in which we have a body of categorical propositions the truth of which is empirically verifiable. When we say that science is becoming more and more mathematical we mean that a mathematical pattern is being increasingly revealed as underlying an array of categorical propositions. It is mostly on account of the complications of the relational structure that simple questions defy simple explanations, in mathematics as well as in science.

The propositions of mathematics are perfectly non-committal as regards the subject matter. In other words, as Bertrand Russell

has remarked, in mathematics 'we do not know what we are talking about'. The propositions of Euclidean geometry remain true, as was pointed out by Hilbert, if the words *point*, *straight line* and *plane* are replaced by *chair*, *table* and *mug* respectively. The mathematical abstractions often masquerade as familiar entities of a world of experience and then their structures and patterns are generally treated as samples of applied mathematics. But this distinction is illogical, for such "uses or applications of mathematics no more constitute a species of mathematics than the uses or applications of a spade constitute a species of spade" (Keyser, 1929).

There is, however, a difference in method. In pure mathematics we proceed from a set of hypotheses to a group of interesting deductions, while in applied mathematics, the relevant hypotheses are to be discovered from the deductions which are already present in the shape of experimental evidence.

In all such explicit analysis there is a neglect of the truth that "there is not a sentence or a word, with a meaning which is independent of the circumstances under which it is uttered... The conclusion is that Logic, conceived as an adequate analysis of the advance of thought, is a fake" (Whitehead, 1947). There is thus a fate of uncertainty hanging over both pure and applied mathematics as both are modes of explicit analysis.

It is well known that there is no finality in the theories of applied mathematics. But it is not generally realized that the validity of a mathematical proof itself is a function of time. There was a time when the proposition could safely be stated that $x^2 = x$ has only two roots. To-day we know the proposition to be wrong if the domain of roots is that of Boolean algebra. Proofs of mathematical propositions that were accepted once are not all valid to-day.

It was G. H. Hardy who wrote, "I have never done anything 'useful'. No discovery of mine has made or is likely to make, directly or indirectly, for good or ill, the least difference to the amenity of

the world." But even he could not get over the fact that he had discovered the law known as Hardy's law in genetics.*

I have mentioned this because Hardy's attitude to applied mathematics is one that is widely prevalent to-day. There is the other attitude of those who go from the trivial to the non-trivial and contribute to our knowledge of Nature as well as to mathematics. In this connection, Wiener (1956), who belongs to the latter group, has some interesting observations to make :

"Thus, here again (in the theory of quasi-analytic functions), as so often in my work, the motivation which has led me to the study of a practical problem has also induced me to go into one of the most abstract branches of pure mathematics.

"This represents an attitude of mine which is in harmony with the greater part of the work of the eighteenth and nineteenth centuries, and which since then has found its representation in the writings of Hilbert and Poincaré. It is not the prevailing attitude at the present time either in America or elsewhere. The two mathematicians now or recently active in America who have adopted a similar point of view are—and I believe not by coincidence—two of the greatest forces in modern mathematics, namely, Hermann Weyl and John von Neumann."

There is an implicit faith that under our perceptions of the physical universe there lies a logical structure. "One should study mathematics", said Birkhoff†, "because it is only through mathematics that Nature can be concieved in harmonious form."

For realizing the harmonies and for gaining a more fundamental control over the forces of Nature it is required that natural processes be represented as logically mechanized. The mechanization has to be done through an appropriate mathematical machine. If the

* *Obituary Notices of Fellows of the Royal Society*, 1948-49, 6, 449.

This is rather a bold expression of humility by one who specialized in the art of building logical structures for the honour of the human spirit.—V. V. N.

† *Bull American Math. Soc.* 1946, 52, 388.

machine works successfully correct predictions of events can be made under circumstances properly controlled. As more and more knowledge is gained in a particular field of phenomena, a stage soon comes when the mathematical machine fails. A new machine has now to be devised. Thus an exciting game goes on between Nature on one side and the mathematician and the scientist on the other. The mathematician invents so that the scientist may discover.

But Nature does not always present problems in a form ready for mathematical applications. Nor have students of Nature always welcomed the use of mathematics in their studies. Comte went as far as to say that every attempt to introduce mathematics into chemistry must be deemed profoundly irrational and contrary to the whole spirit of science.† This is not, of course, the wisdom of the ages for we have the ancient aphorism that "the Book of Nature is written in the characters of Geometry".

There are many natural phenomena such as those considered by D. W. Thomson in his great book, 'On Growth and Form', where the underlying mathematical pattern is quite obvious. But where more profound problems are concerned a greater creative effort is needed, although the crucial steps in the intellectual adventure are just the same as those picturesquely described by Synge (1951), viz.

"(i) A dive from the world of reality into the world of mathematics.

(ii) A swim in the world of mathematics.

(iii) A climb from the world of mathematics back into the world of reality, carrying the prediction in our teeth".*

† D'arcy W. Thompson : *Growth and Form* (1948), Cambridge, p. 11.

* *Math. Gaz.* 1951, 35, 129.

The same ideas are expressed in a more elaborate manner by R. S. Burington in a paper entitled, "On the Nature of Applied Mathematics" in *American Math. Monthly*, 1949, 56, 239.—V. V. N.

How a programme like this is successful is a mystery. "The fact that changes in our material universe can be predicted—that they are subject to mathematical law—is the most significant thing about it, for a mathematical law is a concept of the mind and from the existence of the mathematical law we infer that our minds have access to something akin to themselves that is in or behind the universe" (Whitehead)†. The astonishing fact is that the scientist and the mathematician can collaborate and establish isomorphisms between thought and Nature. That Nature is mathematical was known to the Greeks. "The complementary fact", observed Birkhoff (1950) in 1943, "that mathematics is *natural* is, however just beginning to be grasped ... These two aspects are as intimately related as are the two sides of a single coin".

The two aspects are illustrated firstly by Volterra's study of ravenous populations of sharks and codfish, coexisting as predators and preys, which led to new developments in integro-differential equations and the mathematics of the struggle for existence and secondly, by the mathematical theories of Dirac and Yukawa which led to the discovery of the positron and the meson, after their existence was theoretically predicted.

I shall now describe three mathematical situations : the first will be from the theory of wave equations, the second from the theory of quantum operators and the third from astrophysics. In the first case new mathematical developments are needed. In the second case the intervention of the mathematician has led to a revision of the whole theory. In the third case a problem which applied mathematicians were trying to solve, by the brute force of numercial calculations and calculating machines, was decisively solved by using an effective and elegant mathematical procedure.

In his Gibbs lecture of 1945, under the auspices of the American Mathematical Society, Slater (1946) pointed out what new mathematical work had to be done for a general solution of the wave equation, which has been dominating theoretical physics as a

guiding principle. One method of handling the equation is that of eigenfunctions. The wave function is expressed as a vector in a Hilbert space or as an expansion of the eigenfunctions which satisfy the condition of orthogonality. This method, however, breaks down in most cases as a separation of variables is not generally possible. There are only eleven sets of coordinates in which separation can be effected in three dimensions and they suit only special boundaries. The approximate methods used by physicists for complicated boundary conditions are not generally powerful enough. "No problem", said Slater, "which the mathematician could attack would be more useful to the physicist at present than a real advance in the solution of the non-separable wave equation".

An alternative to the method of eigenfunctions is that of Green's function, which is a solution of the wave equation when it is homogeneous everywhere except at a point. By superposing the solutions one can thus build up an integral equation satisfied by the wave equation whose solution is required. It is not, however, always easy to compute the solution by this method. In solving the problems of wave mechanics it is the method of eigenfunctions that is generally used. For the Schrödinger equation of particles the difficulties are aggravated by the presence of $3n$ coordinates instead of 3. The method of Green's function has not been seriously tried for this problem. It is for the mathematician to find out if some new advances in mathematics and in wave mechanics are not possible by this method.

There are several practical applications in which non-linear partial differential equations arise and for want of an adequate mathematical theory one has to rely on computational devices in handling them. Like the experimental worker who has always to devise new instruments and techniques of observation the theoretician is ever in search of new concepts and algorithms. In some part of mathematics, neglected as a morbid growth by the mathematicians themselves, he may come across the methods and concepts that answer his difficulties ; and like powerful telescopes and microscopes they may reveal new secrets of Nature.

It is well known how a mathematical revolution arose in quantum physics about thirty years ago. It brought harmony and order where contradictions or disorder prevailed either on account of the role of the correspondence principle, or on account of the dual nature of light, or on account of the coexistence of unquantized (aperiodic) and quantized (periodic) motions. Almost simultaneously appeared the matrix mechanics of Heisenberg, Born and Jordan and the wave mechanics of Schrödinger. The transformation theory, developed by Dirac and Jordan, provided a synthesis of the two methods. Use is made in this of improper or 'pathological' functions like the δ-function. The method implies a contradiction in the use of the δ-function and "adheres to the fiction that each self-adjoint operator can be put in diagonal form" (von Neumann, 1955). The procedure that is followed in Dirac's theory amounts to an unjustifiable use of differential operators as integral operators. In order to steer clear of the mathematical difficulties the transformation theory in quantum mechanics has to be rewritten on the lines of the theory of linear operators in the abstract Hilbert space as shown by von Neumann. A central concept of the revised theory is the resolution of the identity corresponding to the operator concerned. It is worthy of notice that the foundations of the theory in the abstract Hilbert space were laid down by E. H. Moore, the originator of general analysis, long before the physicist felt the need, although some of the results, important in the quantum theory, were obtained after the publication of the Dirac-Jordan theory.

In astrophysical studies a differential equation known after Emden† plays a very important part. Eddington has shown that of the polytropic gas models in equilibrium, satisfying the equation, the case $n = 3$ corresponded to the perfectly gaseous stars. For them there is a theoretical relation connecting mass and luminosity well supported by empirical evidence. From purely *a priori* considerations Milne argued that stellar configurations might exist with

† $\frac{1}{\xi^2} \frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) + \theta^n = 0.$

arbitrary sources of energy for which the mass-luminosity law did not hold good. This would be possible for non-gaseous stars satisfying Emden's equation for values of n other than 3. It became therefore necessary to investigate such solutions. This is a tough mathematical problem. What an astrophysicist could do was to carry out the necessary numerical integration either from inside outwards or from outside inwards. Fortunately E. A. Milne brought the problem to the notice of R. H. Fowler who "at once devised a beautiful and at the same time rigorous method of handling and classifying the solutions of Emden's equation for different exponents and for all types of boundary conditions" (Milne, 1945). In the Eddington-Milne controversy of the internal constitution of stars, these results played a decisive part in helping to classify the gaseous and the partly-gaseous configurations. Participating in a debate on stellar structure, G. H. Hardy made the remark that Fowler's work, being pure mathematics, would still be of interest long after the physical theories had become obsolete.

It is interesting to note here how the refinements of measurement have altered the role of mathematics in the theories of the physical world. There was a time when any discrepancy between theory and experiment was attributed to the imperfections of measurement. The theory being mathematical was exact and infallible. But now with the refined modes of measurement it is generally the mathematical theory that is found to be wrong, whenever there is a discrepancy beyond the probable error of measurement. Thus with a discrepancy of the order of 43″ per century, in the motion of the perihelion of Mercury, the necessity arises for replacing the Newtonian theory by general relativity. The detection of certain double lines in the spectrum of an atom requires the introduction of the concept of the electron spin and the spinor theory.

In all such cases the principal criterion is to introduce a new concept so as to account for new facts causing the least possible change in the existing theory. From one base to another the flight of intelligence has to be on the wings of experience.

The existing mathematical theory may be cast into many forms, each a transform of the other. A problem in dynamics may be represented as a problem in partial differential equations and a problem in wave mechanics may be transformed into a problem of calculus of variations. The criterion of the least possible change in the existing theory does not therefore always work effectively in suggesting the new concept.

We have a number of unified field theories starting from Weyl's and ending with Einstein's final attempt. A generalization of general relativity is attempted by introducing some new concept or concepts. A successful generalization is yet to be achieved.

The concept of the cosmological constant λ first appeared in an obvious generalization of Einstein's law of gravitation. It seemed necessary for constructing a model of a finite and bounded universe and for explaining Hubble's law of the redshift of the nebulae. Later developments show that no special purpose is served by the introduction of this concept in the theory.

We can mention any number of concepts which have been successful and which have led to new mathematical formulations and new successful predictions. To name but a few there are the concepts of (1) probability waves, (2) a fundamental particle, (3) an anti-particle and (4) mass and charge renormalization which have brought new mathematical techniques into physics.

When deductive and inductive movements of thought are in progress about the same concepts we often witness, in the no-man's land between pure mathematics and applied mathematics, many interesting results. One day they may be claimed with equal enthusiasm by workers in both the camps, like Levi-Civita's work in the absolute differential calculus.

The evolution of mathematical concepts is a co-operative phenomenon in what it owes to the stream of mathematical culture. Conferences like this, mutual contacts and collaboration have therefore an important part to play in the development of

mathematics. But there is also the 'terribly individual' contribution of a path-breaker like Einstein in the formulation of a concept.†

A concept, successful up to a certain stage, may have to be given up for a new one. Each concept has a limited life 'in the coherent and growing something which we call Mathematics'. A familiar example is of the emergence of the group concept in mechanics and its use in relativity.

We are all familiar with the law of inertia according to which every body perseveres in its state of rest or of uniform rectilinear motion in the absence of impressed forces. The law implies an ambiguity which conflicts with the postulates of an absolute space and an absolute time. It is like the ambiguity with which Buridan's ass was confronted. With a heap of corn on the left and an equal heap on the right, at the same distance, the ass could not find a sufficient reason for turning first to the right or to the left. The principle of sufficient reason left no alternative but to starve to death like a martyr, trotting along the right bisector of the line joining the heap.

There is no sufficient reason why under the law of inertia some bodies should remain at rest and some should move uniformly in a straight line. The concept of a group can be used to describe the exact nature of an ambiguity. In this case it turns out that the Newtonian laws are invariant with respect to the group of Galilean transformations. The law of inertia holds good for an infinity of space-time frames called inertial frames which are transformable into each other by this group of transformations.

The laws of electrodynamics are not, however, invariant with respect to the group of Galilean transformations. The invariance here is with respect to the Lorentz group of transformations which reduce to Galilean transformations, when the parameter representing the velocity of light in free space becomes infinite. It was

† A fine account of the origin and growth of mathematical concepts by Professor R. L. Wilder will be found in *Bull. American Math. Soc.* 1953, *59*, 423-448.

Einstein's genius to show that the Lorentz group is applicable to both mechanics and electrodynamics. Special Relativity deals with both types of phenomena. It gave rise to the replacement of absolute space and absolute time by a combination of the two, space-time, with an invariant metric.

The group was further generalized for general relativity. The laws of general relativity are invariant with respect to the Gaussian group of continuous transformations of coordinates. Under the Lorentz group the Maxwell equations are the simplest invariant laws that an antisymmetrical tensor field can satisfy ; and under the Gaussian group the field equations of general relativity are found to be the simplest invariant laws that a symmetric tensor can satisfy.

Thus we see how the concept of a group, arising out of the ambiguity in Newton's First Law, takes control of electromagnetic and gravitational phenomena separately. Is it possible to generalize the group further so that the simplest laws invariant with respect to it would give the interaction of gravitational and electrodynamic fields ? We have yet to solve the macroscopic relativistic problem of two bodies, each carrying a mass and a charge, moving under the dual forces of gravitation and electrodynamics. The famous deflexion of stellar light in the field of the sun has already shown that an interaction of the two fields exists.

For this purpose a generalization of the group of continuous transformations was tried by increasing the number of dimensions and by the use of complex transformations. This has not succeeded so far.

Newton's famous dictum, viz. '*hypotheses non fingo*' ('I do not make hypotheses') implies that it is for the facts of experience to dictate and not for the mathematician to lay down arbitrarily the central mathematical concept of the theory.

For the unification of the two fields the concept of a generalized world-function has also been tried. The Hamiltonian principle was generalized for fields by Hilbert in 1915 as a world-function principle.

How and when an appropriate concept will emerge, leading to a successful synthesis of the old and the new, none can say. Archimedes hit upon the law of floating bodies when in a bath tub. Newton got the idea of universal gravitation when he saw an apple fall in an orchard. On a geological excursion, while boarding an omnibus, Poincaré made one of his greatest discoveries that the transformations he had used for defining Fuchasian functions were indentical with those of non-Euclidean geometry. The psychology of invention in the field of mathematics is a fascinating subject about which very little is known to-day.

Take, for example, the concept of the field which has played the central role in Einstein's mathematical formulations of physical reality. Its genesis may be attributed to the profound impression made on Einstein's mind, when he was a child of four or five, by the obstinate behaviour of a compass needle, always pointing to the north, wherever it was placed.

There are still several mathematical problems to be solved in the field theory of gravitation.

When two bodies like the sun and the earth are in motion, it is not yet clear from the relativistic description of the motion, whether gravitational waves exist or do not exist. A rigorous mathematical theory of the phenomenon has not been published so far.*

In classical mechanics there is a precise description of the relative motion of two gravitating bodies heading towards each other for a collision. The acceleration changes fast as the relative distance becomes small. In general relativity, the corresponding situation implies a rapid change of the field. Einstein and his co-workers have shown in a series of papers, beginning from 1938, how gravitational motion can be deduced when the field is changing slowly. But for the simple gravitational situation of two bodies, rushing towards each other, the necessary mathematical technique of deriving the

* The latest publication on this subject that I know is J. N. Goldberg's paper on 'Gravitational Radiation' in *Phys. Rev.* 1955, 99, 1873.

equations of motion has not yet been developed. The solution of the problem of the falling apple is still 'in the lap of the gods' (Narlikar and Rao, 1956, *a* and *b*).

There is a corresponding problem in classical electrodynamics where the equations of Maxwell and Lorentz are valid only if the accelerations of the electrons are small. The difficulties associated with the infinite self-energy of the electron are attributed by Dirac to the classical theory. His programme is to revise the classical theory so as to account for rapidly changing fields and then quantize the revised theory. He has introduced the concept of a continuous stream of electricity in which discrete electrons appear as a quantum phenomenon. We have here one of the most important of the unsolved problems of modern physics, viz. the theoretical determination of the dimensionless number, the fine-structure constant. It is a mathematical mystery in a physical setting. Dirac (1954) observes: "It seems hopeless to attack this problem from the physical point of view, as one has no clue to what new physical ideas are needed. However, one can be sure that the new theory must incorporate some very pretty mathematics, and by seeking this mathematics one can have some hope of solving the problem. I believe that the required mathematics will be more closely connected with the equations that describe continuous electric streams than those at present used for point charges."*

In general relativity also there is a difficulty about the particle representation which implies an infinite density concentration at a point. From the classical point of view the gravitational potential energy of such a concentration is negative and infinite. In a rigorous relativistic field theory of 'here-thus relations' such particles with their associated singularities have no place. It was Einstein's hope to obtain non-singular solutions of field equations representing gravitational situations. The unified field theory programme was

* This reminds one of Hilbert's remark : " Actually physics is too difficult for physicists "; I find the remark quoted in Carl Seelig's *Albert Einstein* (1956), Staples, p. 165.

intended partly to remove this deficiency of singularities as field sources in the gravitational theory.

The reconciliation of general relativity with the quantum theory is a difficult mathematical project of which hardly any clues are in sight. The philosophy of physical reality which underlies the field programme of general relativity clashes with the philosophy of the quantum theory with its principle of complementarity. Where atomic phenomena are concerned, 'any sharp separation between the behaviour of atomic objects and their interaction with the measuring instruments is impossible'; whereas in the field theory of macroscopic phenomena the external world is supposed to be independent of the perceiving object. For a unification of the two theories Menger (1949) has suggested a geometry of lumps and a metric, giving for every positive x the probability that the distance between two points does not exceed x.

One way to proceed to the reconciliation of the two theories is through a common problem of microscopic and macroscopic domains. The problem of two massive charged bodies has to be solved so that the solutions of the gravitational problem, the electrodynamic problem and the atomic problem can be deduced as particular cases. In the treatment of this problem the gravitational constant, the constant of the velocity of light, Planck's constant and the fine structure constant are bound to appear as important features. It is necessary, therefore, to consider all the fundamental constants of Nature that would appear in such a synthesis. Eddington (1946) took up this ambitious task and as a common problem of relativity and quantum mechanics he considered the state of equilibrium of a radiationless self-contained system of the total number of particles in the universe. This is an Einstein universe with zero pressure and temperature. It is also a quantized system in the ground state. Further clarification and reformulation of Eddington's ideas are necessary in the light of the critical remarks of Bastin and Kilmister (1954*a*). Moreover, many new phenomena such as the Lamb effect and several fundamental particles have

recently come to light, of which Eddington could not have any information. Peierls (1955) concludes a detailed survey of quantum-mechanical field theory with the following observations:

" However, in this respect we are only at the beginning of the story and it would not be for the first time that a new brilliant idea would bring not only new insight, but a new simplicity into an apparently obscure and complicated situation. "

When the existing theory fails to account for a phenomenon the birth of a new concept is awaited. A critical analysis of the obvious and the concepts which are habits of thought precedes the birth of a new concept.

Such an analysis of the concepts of space-time as a mode of ordering of events is necessary. In the study of quantum phenomena this mode of ordering of events is abandoned. The equations of classical electro-dynamics assume a simple natural form in the relativistic space-time frame. In general relativity there is a further synthesis of space-time and matter. Eddington (1946) tried to deduce the space-time fourfold from the E-operators of his Fundamental Theory. Bastin and Kilmister (1954 *b*) have developed a pre-measurement physical theory as an algebra of facts, propositions and states with the concept of order as a guiding principle. They have an explanation to offer of the four-dimensional structure of the physical universe.

There is no knowing what branch of mathematics may find a structural counterpart in our studies of the physical world. We are familiar with the double-switch arrangement by which the same light can be switched on from one spot and switched off from another. Often, more complicated arrangements are needed, each light being operated by many switches. The problem now arises of designing the most economical arrangement of these switching circuits. C. E. Shannon has discovered that the mathematics involved is a branch of an algebra of logic.

Thus, behind everything that is simple, familiar and commonplace, in life and in Nature, there is a challange to the creative genius in the mathematician. It is a great thing to praticipate in the intellectual adventure of discovering known mathematical patterns in unfamiliar situations and of inventing new patterns for familiar situations. But how often we find in our country, students and teachers of mathematics, either for want of encouragement or for lack of faith and perserverance, turning to other interests and pursuits, after a short and unsuccessful love-affair with Mathematics!

The mediaeval universities aimed to be just repositories of knowledge 'where the scholars turned over together the treasures that the wise had left behind them '. The modern university has to be, in T. H. Huxley's words, a 'factory of knowledge ' as well, where new discoveries and inventions are made. Teachers cannot inspire students unless they have themselves known the thrill of making new contributions to knowledge. As Professor K. Chandrasekharan (1956) rightly points out in his presidential address at the South Asian Conference on Mathematical Education (February 1956) " We are at least fifty years behind the times; what is worse, some of us hardly notice the fact.................The creative scientist has not come into his own, though perhaps the administrative scientist has. This can be rectified only by the assumption of scientific leadership by the universities."

Speaking at the same Conference Professor M. H. Stone (1956) remarked: "...we need to study and reorganize the whole structure of post-graduate instruction in pure mathematics with a view to ensuring that the major results of mathematical research over the last fifty years are brought within the reach of candidates for the Master's degree and within the grasp of candidates for the Doctor's degree."

The mathematization of Natural Philosophy has gone on apace, in recent years, having been particularly accelerated by the exigencies of the war. One can mark the difference in the mathematical levels of 'T and T" or Sir Joseph Larmor's (1900) address on the

methods of mathematical physics and 'Morse and Feshback'.* Unless the standard of mathematical instruction is considerably raised our research students cannot participate effectively in the first-rate work that is being done in applied mathematics these days.

Wide is the gap that exists between the present standards of mathematical teaching and research and what they ought to be. We have to cover it as fast as we can. Let us hope that our Conference will be a source of strength and inspiration to us as we turn to the great tasks that lie ahead.

* 'T and T′' was once the popular name of the treatise on 'Natural Philosophy by Thompson and Tait (1883). The other book is 'Methods of Theoretical Physics by Morse and Feshback (1953).

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ABSTRACTS

ALGEBRA

M. S. HUZURBAZAR, Bombay : *On strongly regular rings.*

A ring R is called *strongly regular* if for every $a \in R$, there exists $x \in R$ with $a = a^2x$. Arens and Kaplansky have shown from topological considerations, that a strongly regular ring has only two-sided ideals. [*Trans. American Math. Soc.* 63, 1948]. In this note we offer an elementary proof of this result and show that an arbitrary ring R is strongly regular if and only if R has only two-sided idempotent ideals. Equivalently, R is strongly regular if and only if $AB = A \cap B$ for every left ideal A and every right ideal B in R.

M. S. HUZURBAZAR, Bombay : *Eigenvalues of matrices with generalized quaternion elements.*

The algebra Q of generalized quaternion elements is a linear associative algebra over the real number field R, with basal elements $1, \omega, \Omega, \omega\Omega$ and the multiplication table $\omega^2 = p$, $\Omega^2 = q$, $\omega\Omega + \Omega\omega = 0$; $p, q \in R$. It is known that Q is isomorphic either to the division algebra of real quaternions or to the algebra $M_2(R)$ of 2×2 real matrices. In this note we consider matrices over Q, where Q is isomorphic to $M_2(R)$.

Two quaternions a, b are called *similar* if there exists a *non-singular* quaternion x (i.e. x has an inverse) such that $ax = xb$. A quaternion λ is called an *eigenvalue* of a square matrix A if there exists a *nonzero* column vector X with $AX = X\lambda$. If λ is an eigenvalue of A, so is every quaternion similar to λ, so that one can speak of the *eigenvalue classes* of A. We obtain the following results :

(i) Every square matrix has an eigenvalue. λ is an eigenvalue of A if and only if the minimum real polynomial of λ is *not coprime* to the minimum real polynominal of A.

(ii) If λ is an eigenvalue of A, there exists a nonzero row vector Y such that $YA = \lambda Y$; and conversely.

(iii) A has only a *finite* number of distinct eigenvalue classes if and only if A has no real eigenvalues.

In general it is not possible to construct a nonsingular matrix with a given nonzero vector as the first column. However using the isomorphism of $M_n(Q)$ to $M_{2n}(R)$ we show that any square matrix is similar to a triangle matrix and obtain a canonical form of A.

D. R. KAPREKAR, Deolali : *Interesting groups of multidigital numbers.*

D. R. KAPREKAR, Deolali : *A conjecture on primes.*

N. A. KHAN, Aligarh : *The characteristic roots of the product of matrices.*

Let $c(A)$ denote a characteristic root of a real or complex matrix, $\bar{c}(A)$ its conjugate. The following theorems regarding $c(AB)$ have been proved :

THEOREM 1. *Let $c(AB)$ denote an arbitrary characteristic root of AB. Then*

$$|c(AB)| < \left\{\max\left| c\left(\frac{A+\bar{A}'}{2}\right)\right| + \max\left| c\left(\frac{A-\bar{A}'}{2i}\right)\right|\right\} \times$$

$$\times \left\{\max\left| c\left(\frac{B+\bar{B}'}{2}\right)\right| + \max\left| c\left(\frac{B-\bar{B}'}{2i}\right)\right|\right\}. \quad (1)$$

THEOREM 2. *Let A and B be two commuting complex matrices of order $n \times n$. If $S_r'(A)$, $S_r''(A)$, $S_r'(B)$, $S_r''(B)$ are the sums of the absolute values of the elements in the r-th row of $(A+\bar{A}')/2$, $(A-\bar{A})2/i$, $(B+\bar{B}')/2$, $(B-\bar{B}')/2i$, respectively ; and if $S'(A)$, $S''(A)$, $S'(B)$, $S''(B)$ are the greatest of the $S_r'(A)$, $S_r''(A)$, $S_r'(B)$, $S_r''(B)$, then*

$$\frac{c(AB)+\bar{c}(AB)}{2} \leqslant S'(A)S'(B) + S''(A)S''(B), \quad (2)$$

and

$$\frac{c(AB)-\bar{c}(AB)}{2} \leqslant S'(A)S''(B) + S''(A)S'(B). \quad (3)$$

If we put $B = 1$ in (2) and (3), we get the results of W. V. Parker [*Duke Math. Journal*, 3 (1937).]

C. R. MARATHE, Aligarh : *On the solution of a scalar matrix equation.*

A square matrix Q over the complex field C is quasi-idempotent if there exists a square λ-matrix $F(\lambda)$ such that $Q^r = F(r)$ for all positive integers r. Such matrices have some interesting properties (see Huff, *American Math. Monthly*, 62 (5), 1955, pp 334-39). Quasi-idempotent matrices are of use in the complete solution of a scalar matrix equation.

Let $f(x) = x^{n_0}(x-\alpha_1)^{n_1}(x-\alpha_2)^{n_2}\ldots(x-\alpha_k)^{n_k}$, where α_i are scalars in C, $\alpha_s \neq 0$, $\alpha_i \neq \alpha_j$ for $i \neq j$ and $s, i, j = 1, 2, \ldots, k$. If a square matrix X over C is a complete solution of the equation $f(X) = 0$, then X may be expressed as

$$X = N_0 + \alpha_1 Q_{(n_1)} + \alpha_2 Q_{(n_2)} + \ldots + \alpha_k Q_{(n_k)},$$

where N_0 is nilpotent (or null) Δ of index not greater than n_0 and $Q(n_s)$ are null or quasi-idempotent Δ of index not greater than n_s. Further N_0 and $Q(n_s)$ are orthogonal in pairs, i.e. $N_0Q(n_s) = 0$, $Q(n_s)\,Q(n_t) = 0$ for $s \neq t$.

C. R. MARATHE, Aligarh : *On bounds for characteristic values of two related matrices.*

Let $A = (a_{ij})$, $B = (b_{ij})$ be $n \times n$ complex matrices and let $F = \sum_{r,s=1}^{n} |b_{rs} - a_{rs}|^2$. If α_1, α_2 are, respectively, the least and the greatest of the characteristic values of $\frac{1}{2}(A + A^*)$, β_1, β_2 the least and the greatest of the characteristic values of $\frac{1}{2}(B + B^*)$, where P^* is the conjugate transpose of P, then

$$\alpha_1 - \sqrt{F} \leqslant \beta_1 \leqslant \alpha_1 + \sqrt{F}; \alpha_2 - \sqrt{F} \leqslant \beta_2 \leqslant \alpha_2 + \sqrt{F}. \quad (1)$$

Let A, B and F be as above. Let α_1', α_2' and β_1', β_2' be the least greatest of the characteristic values of $(A - A^*)/2i$ and $(B - B^*)/2i$ respectively. Then

$$\alpha_1' - \sqrt{F} \leqslant \beta_1' \leqslant \alpha_1' + \sqrt{F}; \alpha_2' - \sqrt{F} \leqslant \beta_2' \leqslant \alpha_2' + \sqrt{F}. \quad (2)$$

Results (1) and (2) were obtained in an attempt to solve an interesting problem suggested recently by Turan [*Publicationes Mathematicae*, 4 (1946), pp. 406-9].

C. R. MARATHE, Aligarh : *A note on the characteristic values of products of matrices.*

Let $\{A_i\}$, be a finite or infinite system of complex square matrices all of which are of the same (finite or infinite) order. Then for $A_s \in \{A_i\}$, we have

$$\prod_s c_{\min}(A_s^* A_s) \leqslant |c(\prod_s A_s)|^2 \leqslant \prod_s c_{\max}(A_s^* A_s), \quad (1)$$

where $c_{\min}(A^*A)$, $c_{\max}(A^*A)$ are, respectively, the least and the greatest of the characteristic values of (A^*A) ; A^* is the conjugate transpose of A and $c(A)$ is any characteristic value of A. Result (1) is a generalization of a result due to Roy [*Proc. American Math. Soc.* 5, 1954, 635-38] and Bela Sz-Nagy [ibid. 7, 1956, 1].

P. S. RAU, Tirupati : *On the extremum of* $\Sigma_1^n x_i^m$ *subject to the conditions* $\Sigma_1^n x_i = 0$ *and* $\Sigma_1^n x_i^2 = 0$.

K. SUNDARESAN, Tirupati : *Number of representations of n* (mod *m*) *as a sum of two numbers.*

S. SWETHARANYAM, Annamalainagar : *A generalization of Brun's theorem.*

Viggo Brun has proved the following : Let for positive ξ, $P(\xi)$ denote the number of numbers $n \leqslant \xi$ such that n and $n+2$ are both primes. Then for $\xi \geqslant 3$,

$$P(\xi) < C\frac{\xi(\log\log\xi)^2}{(\log\xi)^2} \text{ where } C \text{ is a positive absolute constant.}$$

In this paper, we obtain a direct generalization of the above result. We prove the

THEOREM. *Let* $0, x_1, x_2, \ldots, x_{m-1}$ *be any set of distinct fixed non-negative integers. Let for any positive* ξ, $Q(\xi)$ *denote the number*

of numbers $n \leqslant \xi$, such that $n, n+x_1, n+x_2, \ldots, n+x_{m-1}$ are all primes. Then for all $\xi \geqslant$ a proper ξ_0,

$$Q(\xi) < D\,\frac{\xi(\log\log\xi)^m}{(\log\xi)^m},$$

where D is a positive constant depending only upon m and two parameters α and β satisfying the condition $0 < \alpha, \beta < 1$; $\alpha/\beta < 1$ and $m\beta e^2 < 1$.

Weaker generalizations of Brun's theorem have already been obtained by Culanovskii [*Daklady Akad. Nauk SSSR*, (N. S.) 63 (1948), pp. 491-494].

S. Swetharanyam, Annamalainagar : *Asympotic expressions for certain type of sums involving the arithmetic functions in the theory of numbers.*

In this paper, we obtain asymptotic expressions for the sums $\sum_{n=1}^{x} (\phi(n)/n^\alpha)$, $\sum_{n=1}^{x} (d(n)/n^\alpha)$, $\sum_{n=1}^{x} (\sigma(n)/n^\alpha)$ and $\sum_{n=1}^{x} (\sigma_k(n)/n^\alpha)$, for all values of α for which the corresponding infinite sums are not convergent. This enables us to determine certain average orders (in the sense of Hardy and Wright [*Introduction to the theory of numbers*, Oxford (1954) p. 263] connected with the functions $\phi(n)$, etc. and these, we prefer to call as the *weighted average orders* of the functions. In fact, these are the average orders of the *weighted functions* $\phi(n)/n^\alpha$, etc. The average orders of the functions $\psi(n)$, $d(n)$ and $\sigma(n)$ have been found out already and the average order of $\phi(n)/n$ has been obtained by Pillai and Chowla [*J. London Math. Soc.* 5 (1930), pp. 25-101] in some other connection. But the results are all quoted here to complete the sequel.

S. Swetharanyam, Annamalainagar : *On sums involving the fractional parts of numbers.*

The following theorems are proved.

Let $\{f(x)\}$ denote the fractional part of x. Then,

THEOREM 1. $\sum_{d=1}^{x} \{f(x/d)\} = (1-\gamma)x + o(x^{1/2})$, *$\gamma$ being the Euler's constant.*

THEOREM 2. $\sum_{d=1}^{x} d\{f(x/d)\} = \left(1 - \frac{\pi^2}{12}\right) x^2 + o(x \log x)$.

THEOREM 3. $\sum_{d=1}^{x} d^k\{f(x/d)\} = \left[\frac{1}{k} - \frac{\zeta(k+1)}{k+1}\right] x^{k+1} + o(x^k)$,

for all $k \geqslant 2$, $\zeta(s)$ being the Riemann Zeta-function.

The proofs of the above theorems are respectively developed from the asymptotic expressions for the sums $\sum_{n=1}^{x} d(n)$, $\sum_{n=1}^{x} \sigma(n)$ and $\sum_{n=1}^{x} \sigma_k(n)$.

S. SWETHARANYAM, Annamalainagar : *Primes in arithmetical progressions.*

It is shown that given an $\epsilon > 0$, it is possible to find a positive number $x_0(\epsilon)$, such that there is at least one prime p, satisfying $x > p > x(1+\epsilon)$, for all $x > x_0(\epsilon)$. In this paper we prove an analogous theorem for the primes in an arithmetical progression.

It is known that there is at least one prime $p \equiv l \pmod{k}$ in the interval $\left(m, m + \frac{m}{(\log m)} r\right)$, for all $m \geqslant M$, where M is dependent upon r and k alone. In this case, given r and k, the value of M is got explicitly and this is illustrated by the two numerical examples worked out in the paper.

ANALYSIS

TEEKARAO DESAI, Hyderabad : *Solution of the parabolic non-linear partial differential equation.*

TEEKARAO DESAI, Hyderabad : *Solution of the hyperbolic non-linear partial differential equation.*

V. L. KANTHAM, Hyderabad : *On the boundedness of solutions of non-linear differential equations.*

In this paper a lemma of R. Bellman is generalized and has been used to derive some criteria of boundedness of solutions of non-linear differential equations.

V. L. KANTHAM, Hyderabad : *Notes on the general uniqueness theorem.*

First we note that the condition of uniqueness of Kamke's general uniqueness theorem can be improved to some extent.

All the known uniqueness conditions of an initial value problem of ordinary differential equations are only sufficient and need not be necessary. The question whether uniqueness holds even when the criteria of uniqueness are not satisfied is answered in the negative. We show that two different solutions can actually be constructed when the assumptions (both original and improved) of Kamke's theorem are violated. This suggests that these assumptions are the best possible of their kind.

K. NARASIMHAMURTY RAO, Bangalore : *Two theorems on unimodular functions.*

M. R. PARAMESWARAN, Madras : *Some product theorems in summability.*

Let A, B be methods of summability and let (A)'s, (B)'s denote transforms of the real sequence $s=\{s_n\}$ by A, B. It has been shown by several authors (Szasz, Pati, Rajagopal and M. S. Ramanujan) that in certain cases the summability method (A) (B) includes the method (A), where we write (A) $[(B)s] = (A)$ $(B)s$. In this paper I prove that the properties (A) $(B) \supset (A)$ *and* $(A) \supset (A)$ (B) hold simultaneously in certain cases. Typical results are :

THEOREM 1. *Let* $p_n \geqslant 0$, $p_0 > 0$ *and let* $s = \{s_n\}$ *be any sequence.*

Let $N_n(p\,;s) = \dfrac{p_0 s_n + p_1 s_{n-1} + \dots + p_n s_0}{p_0 + p_1 + \dots + p_n}$, $n = 0, 1, 2, \dots$ *and*

let

$$t_n = \frac{\lambda_0 s_n + \lambda_1 s_{n-1} + \dots + \lambda_n s_0}{\sum\limits_{n \geqslant 0} \lambda_n} \quad \textit{where} \sum_{n>0} |\lambda_n| < |\lambda_0|.$$

Then $N_n(p\,;s) \to l$ *implies* $N_n(p\,;t) \to l$ *and conversely.*

THEOREM 2. *Let the Borel transform of* $s = \{s_n\}$, *i.e.*

$$B(x, s) \equiv e^{-x} \sum_{n=0}^{\infty} s_n \frac{x^n}{n!}$$

exist for $x > 0$. *Let*

$$t_n = \frac{\lambda_0 s_n + \dots + \lambda_k s_{n-k}}{\lambda_0 + \lambda_1 + \dots + \lambda_k} \; (n = 0, 1, 2, \dots;\ s_n = 0 \textit{ for } n < 0).$$

Then $B(x\,;s) \to l$ *as* $x \to \infty$ *implies that* $B(x\,;t) \to l$ *as* $x \to \infty$.

If $B(x\,;s)$ *is bounded and* $\sum\limits_{n>0} |\lambda_n| < |\lambda_0|$ *then the converse is also true.*

THEOREM 3. *Let* $t_n = \dfrac{\lambda_0 s_n + \lambda_1 s_{n+1}}{\lambda_0 + \lambda_1}$, $(n = 0, 1, 2, \dots)$ *and let* $B(x\,;s)$ *exist for* $x > 0$. *Then, if either* $\mathrm{Re}\,(\lambda_0 + \lambda_1)$ *and* $\mathrm{Re}(\lambda_1)$ *or* $\mathrm{Im}\,(\lambda_0 + \lambda_1)$ *and* $\mathrm{Im}(\lambda_1)$ *have the same sign,* $B(x\,;t) \to l$ *implies* $B(x\,;s) \to l$, *but not conversely.*

M. R. PARAMESWARAN, Madras: *Note on a theorem of Mazur and Orlicz in summability.*

Let m denote the set of bounded sequences and let (m_p) be its subset, where $x = \{z_n\} \in (m_p)$ if and only if $x \in (m)$ and $x_n = 0$ for $n \leqslant p$. Let A be a convergence-preserving matrix and let (A) denote the set of all sequences x such that $Ax \in (c)$, the set of convergent sequences. We have:

I. $(A)(m) = (c)$ if and only if, the range of A over $(A)(m_p)$ is complete under the metric $\|x\| = \underset{0 \leqslant n < \infty}{\text{l.u.b.}} |x_n|$ for some integer p.

Application of this result gives proofs of earlier results of Copping and of the author on reciprocals of matrices, as also the following results :

II. Let A be a reversible, convergence-preserving matrix with $(A) \neq (c)$ and let the columns of the "associated matrix" be bounded. Then $(A)(m) \neq (c)$.

III. The five conditions : (i) $(A) \supset (c)$, (ii) (A) is reversible, (iii) A is of type M, (iv) $(A) \neq (c)$ and (v) $(A)(m) \subset (c)$ form an inconsistent set, but no *four* of them are inconsistent.

III implies that if a reversible method of type M is strictly stronger than convergence then it must sum some bounded, divergent sequences.

T. PATI, Allahabad: *On properties of Fourier constants.*

In a paper read at the last Conference of the Society (Banaras 1955) the author had conjectured that if $\Sigma a_n \cos nt$ be the Fourier series of an even, periodic (period 2π) function $\phi(t)$, where $\phi(t)\, t^{\delta - 1/p} [\log (k/t)]^{\kappa} \in L^p(0, \pi)$, $1 < p \leqslant 2$, $\kappa > 1/q$ $(1/p + 1/q = 1)$, $k > \pi$, then, for every $\delta > 1/p$, $\Sigma\, n^{-\delta} |a_n| < \infty$. He had further shown, with the help of his earlier analysis and the fact that summability $|C, 1/p|$, $1 < p \leqslant 2$, of the Fourier series of a function of Lebesgue class L is a non-local property, that this conjecture would break down if $\delta = 1/p$.

In the present paper the author establishes the following power-series analogue of his conjecture.

THEOREM 1. If $1 < p \leqslant 2$, $f(z) = \Sigma\, c_n z^n$ *is an analytic function regular for* $|z| > 1$, *and* $f(e^{i\theta})\, \theta^{\delta - 1/p} [\log k/|\theta|]^{\kappa} \in L^p\,(-\pi, \pi)$, *where* $\kappa > 1/q$, $k > \pi$, *then, for every* $\delta > 1/p$, $\Sigma\, n^{-\delta} |c_n| < \infty$.

The proof of this depends upon the following result, which seems to be of independent interest, and is, therefore, stated and proved separately as

THEOREM 2. *If* $1 < p \leqslant 2$, $\kappa > 1/q$, $\beta = 1-(\delta - 1/p)$, *and the Fourier series of the even function* $u(\theta)$, *defined over* $(-\pi, \pi)$ *as* $u(\theta) = |\theta|^{1/p-\delta} \times [\log (k/\theta)]^{\kappa}$, *and by periodicity elsewhere, be given by* $\Sigma\, b_n \cos n\theta$, *then* $\Sigma\, n^{-1+\beta q} |b_n|^q < \infty$.

M. S. RAMANUJAN, Madras : *On Hausdorff and quasi-Hausdorff methods of summability.*

Following Lorentz [*Acta Math.* 80 (1948), 167-190 and *Canadian J. Math.* 1 (1949), 305-319] necessary and sufficient conditions for the T-matrix (H^*, μ_{n+1}), which corresponds to the α_0-matrix (H^*, μ_n), to be strongly regular or to belong to the class $\mathfrak{a}$, are determined in terms of the function $x(t)$ which generates the moment constants μ_n. This along with several other known results gives that the Hausdorff T-matrix (H, μ_n) and the quasi-Hausdorff T-matrix (H^*, μ_{n+1}), determined as earlier, both include the method of Borel's exponential means for the class of bounded sequences, provided the function $x(t)$ is continuous to the left at $t = 1$ also. As a particular case of this we get that if a bounded sequence is summable by Borel's method to l then it is summable (C, ϵ) to l for every $\epsilon > 0$.

The class of methods (s^*, μ) is defined in the paper and the conditions, necessary and sufficient, for (s^*, μ) to be either a K-matrix or a T-matrix are obtained in terms of the sequence μ_n defining the method. The summability strength of the method (s^*, μ) is compared with those of the (H, μ_n) and (H^*, μ_{n+1}) methods.

Theorems on the absolute regularity of the (H, μ_n) method for series-to-series transformations and the (H^*, μ_n) methods for sequence-to-sequence transformations are also proved. These may be looked upon as the natural extensions of the earlier results of Knopp and Lorentz [*Archiv der Math.* 2, (1949-50), 10-16] and Hilda Morely [*J. London Math. Soc.* 25 (1950), 168-173] for the Hausdorff methods and of the author [forthcoming issue of the *J. London Math. Soc.*] for the quasi-Hausdorff methods.

S. M. Shah and S. K. Singh, Aligarh:: *On an inequality for a step function.*

S. K. Singh, Aligarh : *A note on entire and meromorphic functions.*

In this note I prove the following theorems:—

Theorem 1. *Let $f(Z)$ be an entire function of positive integral order. Let S denote the set of entire functions $g(Z)$ such that $T(r, g) = o(T(r, f))$, where $T(r, g)$ and $T(r, f)$ denote the Nevanlinna characteristic functions for $g(Z)$ and $f(Z)$; then the exponent of convergence of the zeros of $f(Z) - g(Z)$ is equal to the order of $f(Z)$ except possibly for one $g(Z) \in S$.*

The above theorem is a generalization of a classical theorem of Borel on a-points. We also give in this note a simple proof of the following theorem of S. M. Shah [*Math. Student,* 12 (1944), 67-70].

Theorem 2. *If $f(Z)$ be a meromorphic function of order $\rho \overset{o}{<} 1$, then* $\limsup_{r\to\infty} \dfrac{N(r, a) + N(r, b)}{T(r, f)} \geqslant 1 - \rho$ *for all a and b* $(a \neq b;$ $0 \leqslant |a| \leqslant \infty$, $0 \leqslant |b| \leqslant \infty)$, *$N(r, a)$ having the usual meaning.*

S. K. Singh, Aligarh, and Hari Shankar, Moradabad: *A note on meromorphic functions in the unit circle.*

Let $f(z)$ be memorphic in $|z| > 1$. Let $n(r, a)$ be the number of a-points of $f(z)$ in $|z| \leqslant r \leqslant 1$ $(0 \leqslant |a| \leqslant \infty)$ and let $\bar{n}(r, a)$ be the number of a-points counted once only. Let

$$N(r, a) = \int_{r_0}^{r} \frac{n(t, a)}{t}\, dt, \quad \bar{N}(r, a) = \int_{r_0}^{t} \frac{\bar{n}(t, a)}{t}\, dt, \quad (0 < r < 1)$$

and

$$\limsup_{r\to 1} \frac{T(r, f)}{\log 1/(1-r)} = \alpha, \quad (0 \leqslant \alpha \leqslant \infty),$$

where $T(r, f)$ is the Nevanlinna characteristic function for $f(z)$. We prove:

Theoerm. $$\sum_{\nu} \left(1 - \limsup_{r\to 1} \frac{\bar{N}(r, a_\nu)}{N(r, a_\nu)}\right) \leqslant 2 + 1/\alpha,$$

where we define $\frac{\bar{N}}{N} = 0$ for $N = \bar{N} = 0$.

We further deduce $\sum_{\nu} \left(1 - \limsup_{\substack{r \to 1 \\ C}} \frac{\bar{n}(r, a_\nu)}{n(r, a_\nu)}\right) \leqslant 2 + 1/\alpha$.

T. P. Srinivasan, Hoshiarpur: *On extensions of measures on a lattice.*

A class F of subsets of a set X is said to be a *lattice* if it is closed for finite unions and finite intersections of sets in it; F is a *ring* if it is a lattice closed for proper differences; F is a *σ-ring* if it is a ring closed for countable unions.

A class F of subsets of X is said to be *compact*, if for each sequence $\{P_n\}$ of sets in F, the relations $\bigcap_{n=1}^{i} P \neq 0$, $n = 1, 2, \ldots$ implies $\bigcap_{i=1}^{\infty}{}^{a} P \neq 0$.

A content is a pair $(\mathcal{U}, \tau)$ where $\mathcal{U}$ is a lattice and τ a non-negitive monotone, finitely additive and finitely subadditive set function on $\mathcal{U}$.

A *measure* is a pair $(\mathcal{U}, \tau)$, where $\mathcal{U}$ is a ring and $(\mathcal{U}, \tau)$ is a content.

Given any classes $\mathcal{U}$, F of subsets of X and a non-negative function τ on $\mathcal{U}$, F is said to *approximate* $\mathcal{U}$ *in the sense* of τ if for every $U \in \mathcal{U}$ and $\epsilon > 0$ there exist $V \in \mathcal{U}$, $P \in F$ with $V \supset P \supset U$ and $\tau(U) < \tau(V) + \epsilon$.

A content $(\mathcal{U}, \tau)$ is said to be *compact* if there exists a compact class F of subsets of X which approximates $\mathcal{U}$ in the sense of τ.

It is easily shown that a compact measure $(\mathcal{U}, \tau)$ is countably additive and has a countably additive extension to the σ-ring generated by $\mathcal{U}$ [E. Marczewski, *Fund. Math.* 40 (1953), 116-117]. In this paper we uphold this result in the case of a compact content by putting an extra assumption on the approximating class F, viz. that $P \in F$, $U \in \mathcal{U}$ implies that $P - U \in F$ and $U - P \in \mathcal{U}$ if $U \supset P$.

Our result generalizes a result of Petis [*Ann. Math.* 54 (1951) p. 193] and covers the standard extension theorem for contents in a locally compact space. The result continues to hold if we replace the assumption of compactness for E by countable subadditivity for τ, and in this form it covers the standard extension theorem for *measures on a ring.*

T. P. SHRINIVASAN, Hoshiarpur : *On representations of linear functionals.*

THEOREM. (*Riesz-Markoff*). *Let C_0 be the class of continuous functions with compact support in a locally compact space X and J, a non-negative linear functional on C_0. Then J has a representation as an integral with respect to a countably additive measure on the minimal σ-ring induced by the functions in C_0, and this representation is unique.*

Our purpose in this paper is to point out that the topological assumptions in the theorem are irrelevant and the theorem is essentially one on linear functionals on a vector lattice of real-valued functions on a set. Specifically we prove the following :

Let X be any set and C_0 a class of non-negative functions on X such that

1. $f, g \in C_0$ implies $f+g, f \cup g, f \cap g \in C_0, f-g \in C_0$ *if* $f_0 \geqslant g$, $\alpha f \in C_0$ *if* $\alpha \geqslant 0$;

2. $1 \cap f \in C_0$ for every $f \in C_0$.

Let J be a non-negative function on C_0 such that

3. for $f, g \in C_0$ $J(f+g) = J(f) + J(g)$, $J(f) \geqslant J(g)$ if $f \geqslant g$, $J(\alpha f) =$ *if* $\alpha \geqslant 0$;

4. for $f_n, f \in C_0$, $f_n \uparrow f$ (pointwise monotone convergence), $J(f_n) \rightarrow J(f)$.

Then the conclusion of Riesz-Markoff theorem is true for J, the uniqueness part being valid in special cases.

Our proof is new even in the classical case.

In this form our result ties up the Stone and Bourbaki theory of integrals as linear functionals with the classical integration theory.

SRIVASTAVA, PRAMILA, Allahabad: *On the properties of the Cesàro means of a function deducible from the summability of the Fourier series generated by it.*

Let $f(t)$ be a periodic function with period 2π and integrable (L) over $(-\pi, \pi)$ and let $\mathscr{S}(f)$ denote the Fourier series of $f(t)$. We write $\phi(t) = \frac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\}$.

As regards the properties of the Cesàro means $\phi_\alpha(t)$ of the function $\phi(t)$ with reference to the summability (C) of $\mathscr{S}(f)$ investigations have been made by a number of authors. The result that if $\mathscr{S}(f)$ is summable (C, α), then $\phi_\delta(t) \to 0$, for $\delta > \alpha + 1$, is the most refined one of its type. With reference to strong summability, Bosanquet [*J. London Math. Soc.* 1932] proved that if $\mathscr{S}(f)$ is summable $(C, 1)$, then $\phi_\delta(t) \to 0$ for $\delta > 1$. Later, he gave another theorem [*Trans. American Math. Soc.* 1936] which, in view of the subsequent definitions introduced by Boyd and Hyslop [*Proc. Glasgow Math. Assoc.* 1952] and by the present author (under publication in the *Proc. National Inst. of Sciences of India*) for summability (R, n, α) and its equivalence with summability (C, α), is to the effect that if $\mathscr{S}(f)$ is summable (C, α), $\alpha > 1$, then $\phi_\delta(t) \to 0$, as t tends to 0, $\delta > \alpha$. In the present paper we extend it to the case $0 < \alpha < 1$, and establish the following

THEOREM. *If* $\alpha > 0$, and $\mathscr{S}(f)$ *is summable* $[C, \alpha]$ *to the sum* $f(x)$ *at* $t = x$, *then* $\phi_\delta(t) \to 0$, *as* $t \to 0$, *for every* $\delta > \alpha$.

It may be remarked that since $(C, \alpha) \subset (C, \alpha + 1)$ this theorem goes deeper than the one for summability (C) quoted above.

SULAXANA KUMARI, Allahabad: *On the behaviour of the Cesàro means of a function and of the Fourier series associated with it.*

1. Suppose that $f(\theta)$ is integrable (L) over $(-\pi, \pi)$ and periodic outside this range with period 2π. Let the Fourier series associated with $f(\theta)$ be

$$f(\theta) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\,\theta + b_n \sin n\,\theta). \tag{1}$$

Writing $\phi(t) = \frac{1}{2}\{f(x+t) + f(x-t) - 2s\}$,

$$\phi_\beta(t) = \frac{\beta}{t^\beta}\int_0^t (t-\mu)^{\beta-1}\,\phi(u)\,du,\ \beta > 0,$$

and denoting by $C_\alpha(\omega)$ the Cesàro means of (1), at $\theta = \omega$, the following theorem in forms (A) and (B) is established.

THEOREM (A). *If for* $\alpha > \rho > -1,\ 1 > \rho > -1, p > -1,$

$$C_\alpha(\omega) - S = o\{\omega^{-\rho}(\log\omega)^p\},\ \textit{as}\ \omega \to \infty, \tag{2}$$

then

$$\phi_\beta(t) = o\{t^\rho(\log 1/t)^{p+1}\},\ \textit{as}\ t \to 0,\ (\beta = \alpha + 1 - \rho), \tag{3}$$

and if for the same range of α *and* ρ

$$C_\alpha(\omega) - S = o\{\omega^{-\rho}(\log\omega)^{-1}\},\ \textit{as}\ \omega \to \infty, \tag{4}$$

then

$$\phi_\beta(t) = o(t^\rho),\ \textit{as}\ t \to 0,\quad (\beta = \alpha + 1 - \rho). \tag{5}$$

(B). *If for* $\alpha > -1,\ 1 > \rho > -1,\ 0 \geqslant \alpha - \rho > -1,\ p > -1,\ \delta > 0,$

$$a_n = O(n^{\alpha-\rho-\delta}),\ b_n = O(n^{\alpha-\rho-\delta}),$$

then (2) *implies* (3), *and* (4) *implies* (5).

The above theorem includes, as particular cases, the results obtained by Loo [*Trans. American Math. Soc.* 56 (1944), 508-518] and by Hyslop [*J. London Math. Soc.* 24(1949) 91-100].

K. VENKATACHALIENGAR and S. K. LAKSHMANA RAO : *Turan's inequality for ultraspherical polynomials.*

Let $P_n^{(\lambda)}(x)$ denote the ultraspherical polynomial and

$$\Delta_n^{(\lambda)}(x) = [P_n^{(\lambda)}(x)]^2 - P_{n+1}^{(\lambda)}(x)\,P_{n-1}^{(\lambda)}(x),$$

$$\Delta_{n,\lambda}(x) = \left[\frac{P_n^{(\lambda)}(x)}{P_n^{(\lambda)}(1)}\right]^2 - \frac{P_{n+1}^{(\lambda)}(x)}{P_{n+1}^{(\lambda)}(1)}\,\frac{P_{n-1}^{(\lambda)}(x)}{P_{n-1}^{(\lambda)}(1)}.$$

Then $\frac{d}{dx}\Delta_n^{(\lambda)}(x)$ is represented as a Wronskian and is shown to have a series representation with terms of constant sign. The sign of $(\lambda - 1)\, x\, \frac{d}{dx^2}\Delta_n^{(\lambda)}(x)$ is determined for general n and λ and that of $\frac{d^2}{dx^2}\Delta_n^{(\lambda)}(\lambda)$ is deduced. The latter expression is also shown to have a positive series representation. $(1 - x)^2\, \Delta_n^{(\lambda)}(x)$ is shown to decrease in $(0, 1)$ for $1 < \lambda \leqslant 3/2$ and inequalities are obtained for $\Delta_{n,\lambda}(x)$ in $|x| \leqslant 1$ of more general nature than the ones known already. When $\lambda = \frac{1}{2}$ (Legendre polynomials), $\Delta_n^{(\lambda)}(x) = \Delta_n(x)$ is shown to have an integral representation, bringing out its positive character in $(-1, 1)$.

APPLIED MATHEMATICS

A. S. APTE, POONA : *Convergence of Fourier series in the application to surface oscillations.*

In the application of the method of Fourier series to practical problems, the rapidity and mode of convergence of the series pose, often, a difficult problem. We have investigated the nature of convergence of Fourier series in the particular case of oscillations on a water surface limited by two rectangular domains with a common boundry. The method is, however, inapt for the formation of derivatives for which methods, much more complicated than the former, have to be resorted to.

P. L. BHATNAGAR and S. K. LAKSHMANA RAO, Bangalore : *General theorems on non-Newtonian viscous flows.*

The study of non-Newtonian fluid motions has been initiated by Rivlin [*Nature*, 160 (1947), 611-613]. The non-Newtonian fluid,

which is highly viscous, is distinguished from the classical viscous fluid by the addition in the stress tensor of second order terms in the components of the rate of deformation tensor. Reiner [*Q. J. M. A. M.*, 1952] puts the stress tensor in the form

$$t^i_j = -p\delta^i_j, + F_1 d^i_j + F_2 d^i_\alpha d^\alpha_j,$$

where p is the mean pressure and $d^i_j = \frac{1}{2}(\nu^i_{,j} + \nu^j_{,i})$.

The constants $\frac{1}{2}F_1 = \mu$, $\frac{1}{2}F_2 = \mu_c$ are the viscosity and cross viscosity coefficients.

In the present paper we discuss some of the general aspects of non-Newtonian flows like the energy equation, circulation variation and the change of vorticity.

(i). We notice that besides the usual expression

$$2\mu[(d^1_1)^2 + (d^2_2)^2 + (d^3_3)^2 + 2(d^1_2)^2 + 2(d^2_3)^2 + 2(d^3_1)^2]$$

the dissipation functions contains an additional term which is a numerical multiple of

$$\mu_c \begin{vmatrix} d^1_1 & d^1_2 & d^1_3 \\ d^2_1 & d^2_2 & d^2_3 \\ d^3_1 & d^3_2 & d^3_3 \end{vmatrix}$$

(ii). The cross viscosity μ_c has no effect on the change of vorticity in plane flows, while this is not true in general for spatial flows.

(iii). We find that the circulation can be conserved in a plane flow if, and only if, the vorticity is harmonic.

(iv). The two plane flows $(u_1, v_1; p_1, V_1)$ and $(u_2, v_2; p_2, V_2)$ are superposable if, and only if, the adjusted pressure π is given by

$$d\left(\frac{\pi}{\rho}\right) = \mu_c d\left[2\left(\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial x} + \frac{\partial v_1}{\partial y}\frac{\partial v_2}{\partial y}\right) + \left(\frac{\partial v_1}{\partial x} + \frac{\partial u_1}{\partial y}\right)\left(\frac{\partial v_2}{\partial x} + \frac{\partial u_2}{\partial y}\right)\right]$$

$$- d(u_1 u_2 + v_1 v_2) + (v_1 \zeta_2 + v_2 \zeta_1)\, dx - (u_1 \zeta_2 + u_2 \zeta_1)\, dy.$$

In order that the resultant motion be uniquely determined from the component motions, it is necessary and sufficient that

$$\frac{\partial}{\partial x}(u_1\zeta_2 + u_2\zeta_1) + \frac{\partial}{\partial y}(v_1\zeta_2 + v_2\zeta_1) = 0.$$

This is the same as the cirterion for the superposability of the classical viscous flows.

VANAJA IYENGAR, Delhi : *On singular static solutions in the Bonnor's modified field of generalized theory of gravitation.*

Einstein and Pauli have shown that in the relativistic theory of gravitation, the field equations do not admit any non-singular static solutions representing a field of non-vanishing total mass. This has been proved also valid in the generalized theory of gravitation with non-symmetric g_{ik}. We have taken the field equations as modified by Bonnor to include the coloumb force between charged particles and shown that the same theorem still holds. The conservation identities are valid for this field and so by calculating the volume integral for the expression for the total energy density of the field giving solutions for large values of r for which the field is weak, and using first approximations we get the value of m, the total mass.

S. I. HUSAIN, Delhi : *On unified field theory of gravitation and elecromagnetism,* I.

In the present paper an attempt has been made to represent the fields of gravitation and electromagnetism as a unified field. The symmetric tensor g_{ij} (interpreted as gravitational potential), the electromagnetic vector ϕ_i, and the non-symmetric coefficients of connection Γ^k_{ij} have been taken as fundamentals and the field equations of the other have been derived by the variational principle taking the scalar density as

$$(-g)^{1/2}(g^{kl} + F^{kl})\, R_{kl} \text{ or } (g^{kl} + F^{kl})\, R_{kl},$$

where F_{kl} is the electromagnetic field vector and R_{kl} the contracted curvature tensor of Γ^k_{ij}. The field equations thus obtained are

$$R_{kl} - \tfrac{1}{2} g_{kl} (g^{ij} R_{ij}) = 0, \tag{a}$$

$$F_{kl} = \phi_{k,l} - \phi_{l,k}, \tag{b}$$

$$F^{kl}_{,l} = 0, \tag{c}$$

where the connexions are given as $\Gamma^k_{ij} = \left\{ {k \atop ij} \right\} + \delta^k_i \phi_j$.

It is noted that the whole set of field equations consists of the purely gravitational field equations (a), to which the system reduces when $\phi_i = 0$, and the Maxwell's equations in free space (b) and (c).

S. I. HUSAIN, Delhi : *On unified field theory of gravitation and electromagnetism*, II.

In the previous paper [I] the connexions of the unified field were defined as $\Gamma^k_{ij} = \left\{ {k \atop ij} \right\} \delta^k_i + \phi_j$. Now we define an affinely connected space to which are assigned these affine connections, and refer to it as U-space. The geometry of U-space is studied in relation to that of Riemannian space whose metric tensor is g_{ij}, and the following two main results have been proved :

(1) The connexions of U-space satisfy the equations $g_{ij/k} + 2 g_{ij} \phi_k = 0$ (solidus denotes covariant diff. w. r. t. Γ^k_{ij}) which is an immediate generalization of an equation satisfied by the connexions of the Riemannian space. Hence U-space is an immediate generalization of Riemannian space.

(2) The connexions of the U-space can be assigned to the Riemannian space so that the geodesics are the paths.

P. C. JAIN, Delhi : *On Chandrasekhar's theory of turbulence.*

S. Chandrasekhar [*Proc. Roy. Soc.* 229A (1955)] has derived a new theory of isotropic turbulence. In this paper, the theory has been generalized to the case of axi-symmetric turbulence, and eight implicit simultaneous equations in eight defining scalars of double and triple correlation tensors have been deduced. Two of these equations are the simplified forms of the equations obtained by S. Chandrasekhar [*Phil. Trans.* (1950)] on the basis of the old theory and thus they replace the equation of Von Karman and Howarth in the theory of isotropic turbulence.

J. N. Kapur, Delhi : *General theory of moderated charges.*

In the present paper, the theory developed earlier by the author [*Proc. Nat. Inst. Sci.* 22A] has been further extended. The conditions for the existence of an equivalent charge in the strict sense have been examined and certain important cases where these are satisfied are obtained. Satisfactory average values of the varying parameters are obtained and equations have been integrated in the most general case. For motion after burnt, it has been shown that an equivalent charge in the strict sense always exists.

J. N. Kapur, Delhi : *An alternative derivation of the energy equation for an R. C. L. Gun.*

The energy equation for an R.C.L. gun was first derived by Corner (1950). An alternative derivation was given by Kothari (1953). In the present paper a third derivation is given. The advantage of the present method is that it makes clear the significances of each term in the energy equation.

J. N. Kapur, Delhi : *The evaluation of co-volume function in Goldie's method of internal ballistics.*

An approximate formula has been given by Goldie to evaluate this function. About the order of the error the statement has been made that the error would be greatest at $f = 0$, where it may be as high as $\pm$ 10%. In the present paper this error has been evaluated in some typical cases and it is found that in general the error can be much higher. The sense in which the error is greatest at $f = 0$ has also been examined.

G. R. Verma, Pilani : *The problem of the twisting of a circular cylinder and that of a paraboloid of revolution by shearing forces on the curved surface.*

This paper deals with the problem of a circular cylinder twisted by equal and opposite forces applied on the curved boundary with the help of two closely fitting circular rings and that of a paraboloid

of revolution in which the end terminating on a confocal paraboloid is fixed and a shearing force is applied along a circular ring on the curved surface of the paraboloid. Dirac's delta function, which is used here, is found to be very useful in solving this type of problems.

P. D. S. VERMA, Kharagpur : *Solid compressible rotating shaft under loading.*

By taking Green's special constitutive equations for plastic flow, the problem of solid compressible rotating shaft under loading has been solved. The results are found to be in perfect agreement with those of Nadai for the incompressible case.

GEOMETRY

MADHUMALATI APTE, Poona : *On the automorphisms of almost hermitian and almost Kählerian manifolds.*

Let V_{2n} be an almost hermitian manifold. We define on this manifold two hermitian connexions, introduced by A. Lichnerowicz and S. S. Chern. We examine under what conditions an isometric transformation or an affine transformation (for one of the two connexions) of V_{2n} is also an automorphism of V_{2n}. We next examine the corresponding results in the case when V_{2n} is an almost Kählerian compact manifold. For such a manifold, every isometric transformation is an automorphism, and if the quadratic exterior form representing Chern's characteristic class of degree 2 is zero, to every Killing vector ξ corresponds a harmonic vector $J\xi$, where J is the operator of the almost complex structure.

GITA HALDAR AND RAM BEHARI, Delhi : *Generalized asymptotic lines in Riemannian manifolds.*

Defining a generalized asymptotic line C in a Riemannian V_n immersed in a V_{n+1}, as a line which is self-conjugate as well as

conjugate to its first $(n-2)$ principal normals with respect to the hypersurface V_n, we obtain some properties of C, including generalizations of Bonnet's formula for geodesic torsion and Beltrami-Enneper formula for asymptotic lines in a V_2.

GITA HALDAR and RAM BEHARI, Delhi : *Generalized asymptotic lines under p-restrictions in Riemannian manifolds.*

Defining a generalized asymptotic line under p restrictions in a Riemannian V_n immersed in a V_{n+1}, as a line which is self-conjugate as well as conjugate to its first $(n-p-2)$, $p > 0$, principal normals with respect to V_n, we obtain some properties of the generalized asymptotic line, under p restrictions which have been illustrated by assigning special values to p.

K. N. KAMALAMMA, Bangalore : *Some properties of Ribaucour congruences.*

Some new properties of Ribaucour congruence and of the generalized Ribaucour congruence defined by omitting the orthogonality condition are set forth in this paper.

In a Ribaucour congruence, if the characteristic ruled surfaces meet the director surface along curves intersecting orthogonally, then the harmonic mean of the principal curvatures is equal to the sum of the normal curvatures along the directions in which the developables meet the director surface.

If θ_1 is the angle between the curves along which the principal surfaces meet the director surface and ω is the angle between the curves along which the developables meet the director surface then they are connected by $2 \tan \theta_1 + \tan \omega \sin \omega = 0$.

If the director surface is a minimal surface, then the principal surfaces meet the director surface along curves intersecting at right angles.

R. N. KAUL, Delhi : *On the magnitude of the derived vector of a unit normal to a hypersurface.*

The magnitude of the derived vector of a unit normal along the tangent vector to a curve in the hypersurface has already been defined by me in a former paper [*Académie Royale de Belgique, classe des sciences*, XLI, 1955, pp. 1292-1304]. The object of this paper is to find a convenient expression for this magnitude of the derived vector so as to enable us to obtain a few known results directly. The extension of the above expression to a subspace in a Riemannian space has also been derived.

R. N. KAUL, Delhi : *Curvatures in Finsler space.*

Consider a curve $C : x^i = x^i(s)$ with curvatures $1/\rho_i$ $(i = 1, 2, 3)$. Denote by $b^i_{h|}$ and $b^{*i}_{h|}$, $(h = 1, 2, 3)$, the set of tangent, principal normal, and the binormal unit vectors at the points P and P^* respectively of the curve. Let θ_h be the inclination of the vector $b^i_{h|}$ and $b^{*i}_{h|}$, then the results

$$\lim_{s\to 0}\frac{\theta_1^2}{s^2} = \frac{1}{\rho_1^2}, \quad \lim_{s\to 0}\frac{\theta_2^2}{s^2} = \frac{1}{\rho_1^2} + \frac{1}{\rho_2^2}, \quad \lim_{s\to 0}\frac{\theta_3^2}{s^2} = \frac{1}{\rho_2^2},$$

where s is the arc length from P to P^* are well known in Euclidean space of three dimensions.

H. Levy has extended the above results to a Riemannian space of n dimensions. The purpose of the present paper is to obtain their extension for a general space with a metric involving the coordinates and the components of the line-element first suggested by P. Finsler.

S. K. KAUL, Delhi : *A generalization of the theorem of Green.*

In this paper I have generalized the Green's theorem of a Riemannian manifold by expressing it in terms of extensors. The definition of a Laplacean of vectors has been extended for extensors, and certain consequential deductions have been made with the use of the generalized Green's theorem. The main results obtained are :

(1) In a compact orientable Riemannian manifold V_n,

$$\int_{V_n} \lambda^{\alpha i};\ \alpha i dv = 0,$$

for any arbitrary excontravariant extensor $\lambda^{\alpha i}$. The ';' denotes the excovariant differentiation.

(2) In a compact orientable Riemannian manifold V_n, for any scalar field ϕ, we have

$$\int_{V_n} \Delta(\phi)\, dv = 0,$$

where $\Delta(\phi)$ is the ex-Laplacean of ϕ.

(3) In a compact orientable Riemannian manifold, if a scalar field ϕ, satisfies $\Delta(\phi) \geqslant 0$ everywhere in the manifold, then ϕ is constant w.r.t. $(x^{(\alpha)i})$'s, and $\Delta(\phi) = 0$ everywhere on the manifold.

S. K. KAUL, Delhi : *On the extensor differentiation of extensors.*

In this paper I have considered extensors in a subspace embedded in a Riemannian space and have developed the theory of "extensor differentiation of extensors". The main result is the deduction on the basis of this theory, of the theorem :

If a curve in a subspace of a Riemannian space is a geodesic relative to the enveloping space it is a geodesic relative to the subspace also.

H. S. NAGARATHNAMMA, Bangalore : *The Darboux and Laguerre functions for a hypersurface in Riemannian space.*

If E and ξ are vectors of a V_{n+1} whose components in the subspace V_n are e^i, ϵ^i respectively, and if G is a contravariant tensor of the second order in V_{n+1}, we define with the usual notations,

$$E.\frac{\delta G}{\delta s}.\xi = a_{\alpha\beta}\, a_{\gamma\delta}\, y^{\alpha}_{;p}\, e^p\, G^{\beta\gamma}_{;i}\, e^i\, y^{\delta}_{;j}\, \epsilon^j.$$

The expression

$$D_{h|h} = E_h.\left(\frac{\delta}{\delta s_l} \nabla N\right). E_h$$

is defined as the Darboux function of the direction e_h w.r.t. the direction e_l, $h \neq l$. When $h = l_i$ this is the Laguerre function for $e_{h|}$. We prove

$$D_{hhh} = \frac{\delta}{\delta s_{h|}} \Omega_{pq} e^p_{h|} e^q_{h|} - 2\, \Omega_{pq} e^p_{h|} p^q_{h|} \cdot \frac{1}{\rho_{g|h}},$$

where $p^q_{h|}/\rho_{g|h}$ is the principal normal vector *of* $e_{h|}$. This formula reduces in ordinary space to the well-known Leguerre function

$$L = \frac{d}{ds} K_n - 2\, T \gamma.$$

We also prove that the sum of the Darboux functions of a direction ϵ w. r. t. any n mutually orthogonal directions of V_n is numerically equal to the intrinsic derivative of the mean curvature of V_n in the direction ϵ.

M. N. Rajalakshamma, Bangalore : *Certain special line-congruences.*

(1) The congruence of lines intersecting a given line and a given conic satisfies a quadratic complex whose equation is expressible in the form

$$(Ap_{23})^2 + (Bp_{31})^2 = (ABp_{34})^2.$$

We deduce that there are six reguli included in the congruence.

(2) The lines joining a given line and a twisted cubic form the intersection of two tetrahedral complexes.

S. C. Saxena and Ram Behari, Delhi : *Generalized Lame's equations in generalized Riemann space.*

In this paper, congruences and orthogonal ennuples have been studied. The curl of a vector has been defined and generalized curl of a vector is also defined in generalized Riemann space. Irrational congruences have been studied and generalized Lame's equations in generalized Riemann space have been deduced.

S. C. Saxena and Ram Behari, Delhi : *Union curves, union curvature and relative parallelism in generalized Riemann space.*

The results of this paper are generalizations of some results obtained by T. K. Pan and C. E. Springer in generalized Riemann space.

The equations for relative parallelism in a generalized Riemann space have been deduced and equations for union curves and pseudo-geodesics have also been deduced.

M. SRIRANGAMMA, Bangalore : *Some properties of the configuration formed by a complete quadrangle and quadrilateral.*

K. SITARAM, Tirupati : *Some properties of the θ-pedal circle.*

STATISTICS

P. C. CONSUL, Khurja : *On the cumulants of the distribution of the sum of squares of dispersion about the mean of n ordered variates subject to a linear restraint.*

M. S. Bartlett [J. L. M. *Soc.* 13, 1938] considered the case of two variables u and u_1 whose joint distribution function is $f(u, u_1)$ and obtained the characteristic function of u when u_1 is fixed.

Utilising the joint distribution of a number of dispersion varieties given by Banerjee [*Proc. Ind. Aca. Sci.* 35 (1), 1952] Bartlett's above principle has been applied in finding the characteristic function of the distribution of the sum of squares of dispersion about the mean of n ordered variates subject to a linear restraint. Then the cumulants and other constants of the distribution have been determined.

P. C. CONSUL, Khurja : *On the sampling distribution of the mean for the distribution of the divergence D'^2 between two multinomial populations.*

A. Bhattacharya [*Sankhya*, 7 (4), 1946] defined a new measure of divergence D' between two samples of sizes n_1 and n_2 drawn from two multinomial populations and obtained the distribution of D'. In an unpublished paper, while studying the properties of the distribution,

I obtained its characteristic function. Utilising that characteristic function in the present paper, I have obtained the sampling distribution of the mean for the distribution of the divergence D'^2 and then have studied its properties with the help of moments and cumulants.

V. S. HUZURBAZAR, Poona : *Explicit sampling distribution of the maximum likelihood estimators of the parameters of a distribution admitting a set of jointly sufficient statistics.*

This paper deals with the extension to the case of several parameters of the corresponding problems in the case of a single parameter considered by Fisher (1934) and Pitman (1936). Explicit posterior probability distribution of the maximum likelihood estimates from a subsequent sample when an earlier sample is known, is also obtained, thus extending the previous result obtained by the present author (1948).

M. V. JAMBUNATHAN, Mysore : *On certain maximal and minimal solutions.*

G. SANKARANARAYANAN, Annamalainagar: *Some asymptotic properties of Poisson stochastic process.*

The Poisson process $X(t, \omega)$ $[\omega \in \Omega,\ 0 \leqslant t < \infty]$ is a temporally and spatially homogeneous Markoff process satisfying

(i) $X(0, \omega) = 0$ and $X(t,\omega) =$ integer $\geqslant 0$ for every $\omega \in \Omega$; and

(ii) $$\Pr\,[X(t, \omega) - X(t', \omega) = i] = \frac{[\lambda(t-t')]^i}{i!}\,e^{-\lambda(t-t')} \text{ for } t > t',$$

where i is a non-negative integer and λ is a positive constant.

Let $L_m(\omega) = t_{m+1}(\omega) - t_m(\omega)$, where $t_m(\omega) = \min\,[T, X(T, \omega) = m]$. It is knwon that $L_0, L_1, L_2, \ldots, L_m, \ldots$ are mutually independent random variables with a common distribution function

$$F(x) = 1 - e^{-\lambda x} \text{ for } x \geqslant 0, \text{ and } = 0 \text{ otherwise.}$$

Let $M_n = \max(L_0, L_1, \ldots, L_{n-1})$ and $L_n = (L_0 + \ldots + L_n)/M_n$. From the sequence $L_0, L_1, \ldots$, we form a new sequence $Y_1, Y_2, \ldots$, where Y_1 is the mean of the first n elements, Y_2 is the mean of the next n elements, and so on. $Y_1, Y_2, \ldots$, will also form a sequence of independent variables.

Let $U_m = \max(Y_1, Y_2 \ldots, Y_m)$ and $l_m = \min(Y_1, Y_2, \ldots, Y_m)$.

We prove the following theorems.

THEOREM 1. *If* $0 < \alpha < 1$, *then*

$$\Pr\left[\liminf_{m\to\infty} \frac{n\lambda U_m}{\alpha \log m} \leqslant 1\right] = 1.$$

THEOREM 2. *If* $\beta > 1$, *then*

$$\Pr\left[\limsup_{m\to\infty} \frac{l_m}{\beta}\left(\frac{n!}{n^n \lambda^n} \frac{\log m}{m}\right)^{-1/n} \leqslant 1\right] = 1.$$

THEOREM 3. *For any* $p > 0$,

$$\Pr\left[\lim_{m\to\infty} \frac{M_n}{(n \log n)^{2+p}} = 0\right] = 1.$$

THEOREM 4. *For any* $p \geqslant 0$,

$$\Pr\left[\lim_{m\to\infty} \frac{L_n}{n^2 (\log n)^p} = 0\right] = 1.$$

THEOREM 5. $(t_1 + \ldots + t_m)/m^2$ *converges in probability to* $1/2\lambda$.

G. SANKARANARAYANAN, Annamalainagar: *Sums of independent randam variables.*

G. Kallianpur and H. Robbins have proved the following results [*Duke Math. Jour.* 21, 1954]:

Let $(X_1, Y_1), \ldots, (X_n, Y_n) \ldots$ be a sequence of independent random vectors with common probability density $\frac{1}{2\pi} e^{-(x^2+y^2)/2}$. Let

$S_n = X_1 + \dots + X_n$, $T_n = Y_1 + \dots + Y_n$. Suppose that $L_i(x, y)$, $i = 1, 2$, are real valued functions bounded and summable in the plane $-\infty < x < \infty$, $-\infty < y < \infty$. Let

$$\bar{L}_i = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} L_i(x, y)\, dx\, dy.$$

Then

$$\lim_{n\to\infty} \Pr\left[\frac{\bar{L}_i \log n}{2\pi} \sum_{j=i}^{n} L_i(S_j, T_j) \leqslant x\right] = G(x),$$

where

$$G(x) = 1 - e^{-x} \text{ for } x \geqslant 0 \text{ and } 0 \text{ for } x < 0.$$

If $\bar{L}_2 \neq 0$, then $\lim_{n\to\infty} \frac{\Sigma L_i(S_j, T_j)}{\Sigma \bar{L}_2(S_j, T_j)} = \frac{\bar{L}_1}{\bar{L}_2}$, in probability, the summation in both the Σ's being from $j = 1$ to $j = n$.

When we consider the corresponding result for three dimensions, the various moments of the corresponding sums all tend to finite limits as n tends to infinity. After having found that the results of Kallianpur and Robbins cannot be directly generalized to higher dimensions, we have investigated the limiting distribution as n tends to infinity of the sums

$$\sum_{j=1}^{n} L\left(W_k(j)\, S_j^{(1)}, W_k(j)\, S_j^{(2)}, \dots, W_k(j)\, S_j^{(k)}\right)$$

properly normalized by taking $(S_j^{(1)}, S_j^{(2)}, \dots, S_j^{(k)})$ in a different manner, $W_k(j)$ being a suitable weight function. These results will have immediate bearing on the Brownian motion process.

TOPOLOGY

R. VENKATARAMAN, Madras : *Lindelöf spaces.*

If θ is a given cardinal, we say that a topological space is θ-compact if every open covering has a subcovering whose cardinal is less then θ. When $\theta = \aleph_0$ or $\aleph_1$ we get the usual compact (bicompact) and Lindelöf spaces respectively. It turns out that most of the characterizations of compact spaces continue to be true for θ-compact spaces also, with suitable modifications.

We find that a regular space is Lindelöf if, and only if, it is an "Ω-closed" subset of the direct product of copies of the closed interval (0, 1).

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NEWS AND NOTICES

MEMBERS of the Indian Mathematical Society who wish to present papers to the twenty-third Conference to be held in December 1957 are requested to send in their papers *in full*, along with *two copies* of an abstract of each paper, in time to reach the Editor, Journal of the Indian Mathematical Society, Tata Institute of Fundamental Research, Apollo Pier Road, Bombay 1, not later than October 1, 1957. The abstracts should be *type-written* on special sheets which may be obtained on request from the Secretary of the Society. No papers and abstracts will be accepted after October 1.

At the invitation of the Polish Academy of Sciences, Professor K. Chandrasekharan participated in an International Colloquium on Harmonic Analysis held at Warsaw on 16-18 May 1957. He also visited the Universities of Krakow and Wroclaw and gave lectures. At the invitation of the Swiss Federal Institute of Technology, Zurich, he gave a course of five lectures on "Functional equations and Dirichlet series". He also visited and lectured at the University of Basle and the Imperial College of Science and Technology, London.

The University of Illinois announces the forthcoming publication of the Illinois Journal of Mathematics, a quarterly devoted to the publication of basic research in pure and applied mathematics. The first number will appear early in 1957, and it is planned that each number will contain about 150 pages.

The present board of editors consists of Reinhold Baer, J. L. Doob, and A. H. Taub, University of Illinois ; George W. Whitehead, Massachusetts Institute of Technology ; and Oscar Zariski, Harvard University.

The Illinois Journal will publish papers in English, French, German, and Italian. Manuscripts should be sent to the Illinois Journal of Mathematics, Department of Mathematics, University of Illinois, Urbana, Illinois, or to one of the editors.

The Illinois Journal of Mathematics is made possible by a grant from the bequest to the University of Illinois by the late Professor G. A. Miller, and by an initial subvention from the American Mathematical Society.

The subscription price is $9.00 a volume (four numbers), with a special rate of $5.00 for individual members of the American Mathematical Society. Subscriptions should be sent to the University Press, 207 Administration East, University of Illinois, Urbana, Illinois, U.S.A.

[*Added in proof.* The first issue of the Illinois Journal of Mathematics has appeared. Contents : *A fundamental inequality in the theory of extension of valuations* by I. S. Cohen and Oscar Zariski; *Kompakte projective Ebenen* by Hans Freudenthal; *Homology of Noetherian rings and local rings* by John Tate ; *Finite dimensionality of certain transformation groups* by Deane Montgomery; *On modules of trivial cohomology over a finite group* by Tadasi Nakayama ; *Markoff processes and potentials* by G. A. Hunt; *Inequalities for asymmetric entire functions* by R. P. Boas Jr. ; *On a class of linear differential equations with periodic coefficients* by Jack K. Hale ; *On a problem of Picard concerning symmetric compositums of function fields* by Jun-Ichi Igusa; *On matrix classes corresponding to an ideal and its inverse* by Olga Taussky.

The issue runs to 113 pages and is well printed.]

The Third Congress on Theoretical and Applied Mechanics will be held under the presidentship of Dr. S. R. Sen Gupta on December 24-26, 1957 at the Indian Institute of Science, Bangalore. Papers intended for the Conference should be forwarded, along with three copies of an abstract of each paper, to Professor B. R. Seth, Indian Institute of Technology, Kharagpur, so as to reach him by September 30, 1957. Further information may be obtained from Professor B. R. Seth.

International Congress of Mathematicians 1958
First Communication

At the invitation of the City and University of Edinburgh and the Royal Society of London, the International Congress of Mathematicians will meet in Edinburgh from August 14th to August 21st, 1958. His Royal Highness the Duke of Edinburgh has graciously consented to extend his patronage to the Congress.

The Executive Committee is inviting a number of mathematicians to deliver one-hour and half-hour addresses. There will also be daily sessions devoted to fifteen-minute communications. Those who wish to present such communications will have an opportunity of offering to do so when they receive the Second Communication.

There will be eight sections, namely :—

1. Logic and Foundations.
2. Algebra and Theory of Numbers.
3. Analysis.
4. Topology.
5. Geometry.
6. Probability and Statistics.
7. Applied Mathematics, Mathematical Physics and Numerical Analysis.
8. History and Education.

A programme of entertainments and excursions is being planned. There will be two categories of membership of the Congress :

Ordinary members, who will be entitled to participate in the scientific and social activities of the Congress and to receive the Proceedings of the Congress, and

Associate members accompanying ordinary members of the Congress, who will be entitled to many of the privileges of ordinary

membership, but will not participate in the scientific programme and will not receive the Proceedings.

Those who wish to receive further information about the Congress are requested to communicate their name and full address to the Secretary : Frank Smithies, Mathematical Institute, 16 Chamber Street, Edinburgh 1, Scotland, as soon as possible. Towards the end of 1957 they will receive a Second Communication giving more detailed information and specifying the fees to be paid by members; a form of application for membership of the Congress will be included.

It would be of considerable assistance if, where possible, correspondents would use English, French, German or Russian.

ON THE COEFFICIENTS OF AN ENTIRE SERIES OF FINITE ORDER

By Q. I. RAHMAN

1. Let $f(z) = \sum_0^\infty a_n z^n$ be an entire function of order $\rho (0 < \rho < \infty)$, and let

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log M(r)}{r^\rho} = \frac{T}{t};\ \lim_{n\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \left(\frac{1}{\rho e} n|a_n|^{\rho/n}\right) = \frac{\phi}{\theta}.$$

It is known [1, pp. 44-46; 3, pp. 45-46] that $t = \phi$ and $t \geqslant \theta$, with equality sign for the class of functions for which $|a_n/a_{n+1}|$ is a non-decreasing function of n for all large n.

In this paper I compare $\log M(r)$ with $r^\rho (l_1 r)^{\alpha_1} (l_2 r)^{\alpha_2} \ldots (l_k r)^{\alpha_k}$ where $l_k x$ denotes the kth iterate of $\log x$ [2, p.16] and obtain relations which include the above mentioned results as particular cases. I also consider a class of integral functions of zero order and define the type of such functions suitably and obtain analogous results for this class of functions.

2. Let

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log M(r)}{r^\rho\, (l_1 r)^{\alpha_1} \ldots (l_k r)^{\alpha_k}} = \frac{T(\rho)}{\tau(\rho)};$$

$$\lim_{n\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \left(\frac{\rho^{\alpha_1 - 1}}{e} \cdot \frac{n}{(l_1 n)^{\alpha_1} \ldots (l_k n)^{\alpha_k}} |a_n|^{\rho/n}\right) = \frac{\vartheta}{\odot};$$

$$\lim_{n\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \left\{\frac{(\alpha_1 - 1)^{\alpha_2}}{\alpha_1^{\alpha_1}} \cdot \frac{n^{\alpha_1}}{(l_1 n)^{\alpha_2} \ldots (l_{k-1} n)^{\alpha_k}} \left(\frac{\alpha_1 - 1}{\log 1/|a_n|}\right)^{\alpha_1 - 1}\right\} = \frac{u}{v}.$$

THEOREM I. *If* $f(z) = \sum_0^\infty a_n z^n$ *be of order* $\rho (0 < \rho < \infty)$, *then*

(i) $\tau(\rho) = \vartheta$,

(ii) $\tau(\rho) \geqslant \odot$,

(iii) *if further* $|a_n/a_{n+1}|$ *is a steadily increasing function of* n *for all large* n *then* $\tau(\rho) = \odot$.

PROOF. (i) Suppose first that $0 < \vartheta < \infty$ and let n_0 be such that $l_k n_0 > 0$. Then $\dfrac{n}{(l_1 n)^{\alpha_1}\ldots(l_k n)^{\alpha_k}} \, |a_n|^{\rho/n} > (\vartheta - \epsilon) \dfrac{e}{\rho^{\alpha_1 - 1}}$ for a sequence of values of n tending to infinity, say, $n_1, n_2, \ldots, n_p, \ldots (n_1 > n_0)$. By Cauchy's inequality $M(r) \geqslant |a_n| \, r^n$, and so for $n = n_p$,

$$\frac{\log M(r)}{r^\rho (l_1 r)^{\alpha_1}\ldots(l_k r)^{\alpha_k}} \geqslant \frac{\log |a_n| + n \log r}{r^\rho (l_1 r)^{\alpha_1}\ldots(l_k r)^{\alpha_k}}$$

$$> \frac{\frac{n}{\rho}\log\left\{(\vartheta - \epsilon) e/\rho^{\alpha_1 - 1}\right\} - \frac{n}{\rho}\log\left\{\frac{n}{(l_1 n)^{\alpha_1}\ldots(l_k n)^{\alpha_k}}\right\} + n \log r}{r^\rho (l_1 r)^{\alpha_1}\ldots(l_k r)^{\alpha_k}}.$$

Let

$$A = \frac{\rho^{\alpha_1 - 1}}{\vartheta} \text{ and } r = r_{n_p} = \left\{\frac{n_p A}{(l_1 n_p)^{\alpha_1}\ldots(l_k n_p)^{\alpha_k}}\right\}^{1/\rho}, \ (p = 1, 2, \ldots).$$

Then

$$l_1 r_{n_p} = (1/\rho)\, l_1 n_p + o(l_1 n_p);\ l_2 r_{n_p} = l_2 n_p + o(l_2 n_p), \text{ etc.}$$

and so, for $n = n_p$ and $r = r_{n_p}$,

$$\frac{\log M(r)}{r^\rho (l_1 r)^{\alpha_1}\ldots(l_k r)^{\alpha_k}}$$

$$> \frac{\frac{n}{\rho}\log\left\{\frac{(\vartheta - \epsilon)e}{\rho^{\alpha_1 - 1}}\right\} - \frac{n}{\rho}\log\left\{\frac{n}{(l_1 n)^{\alpha_1}\ldots(l_k n)^{\alpha_k}}\right\} + \frac{n}{\rho}\log\left\{\frac{nA}{(l_1 n)^{\alpha_1}\ldots(l_k n)^{\alpha_k}}\right\}}{\left\{\frac{nA}{(l_1 n)^{\alpha_1}\ldots(l_k n)^{\alpha_k}}\right\}\left\{\frac{1}{\rho} l_1 n + o(l_1 n)\right\}^{\alpha_1}\ldots\left\{l_k n + o(l_k n)\right\}^{\alpha_k}}$$

$$= \frac{\frac{n}{\rho}\log\left\{(\vartheta - \epsilon)\, e/\rho^{\alpha_1 - 1}\right\} + \frac{n}{\rho}\log A}{\left\{\frac{nA}{(l_1 n)^{\alpha_1}\ldots(l_k n)^{\alpha_k}}\right\}\frac{1}{\rho^{\alpha_1}}(l_1 n)^{\alpha_1}\ldots(l_k n)^{\alpha_k}\left\{1 + o(1)\right\}}$$

$$= \frac{\rho^{\alpha_1 - 1}}{A} \cdot \frac{\log\{(\vartheta - \epsilon)\, eA/\rho^{\alpha_1 - 1}\}}{\{1 + o(1)\}}$$

$$\sim \vartheta \log\{(\vartheta - \epsilon)\, e/\vartheta\}.$$

Hence $\tau(\rho) \geqslant \vartheta$, which obviously holds when $\vartheta = 0$. If $\vartheta = \infty$, the above argument shows that $\tau(\rho) = \infty$.

To prove that $\vartheta \geqslant \tau(\rho)$ we may suppose $\tau(\rho) < \infty$. If $f(z)$ is an entire function of order $\rho (0 < \rho < \infty)$, then $\log M(r) \sim \log \mu(r)$ and therefore for a sequence of values of r, say, $r_1, r_2, \ldots, r_p, \ldots$

$$\log \mu(r) > \{\tau(\rho) - \epsilon\}\, r^{\rho}\, (l_1 r)^{\alpha_1} \ldots (l_k r)^{\alpha_k}. \tag{1}$$

Let $\mu(r_p) = |a_{n_p}|\, r_p^{\,n_p}$, then for $r = r_p$ and $n = n_p$, we have

$$n \log r + \log |a_n| > \{\tau(\rho) - \epsilon\}\, r^{\rho} (l_1 r)^{\alpha_1} \ldots (l_k r)^{\alpha_k}.$$

If

$$X = \frac{n}{(l_1 n)^{\alpha_1} \ldots (l_k n)^{\alpha_k}}\, |a_n|^{\rho/n}, \tag{2}$$

then for $r = r_p$, $n = n_p$,

$$\log X = \log \left\{ \frac{n}{(l_1 n)^{\alpha_1} \ldots (l_k n)^{\alpha_k}} \right\} + \frac{\rho}{n} \log |a_n|$$

$$> \log \left\{ \frac{n}{(l_1 n)^{\alpha_1} \ldots (l_k n)^{\alpha_k}} \right\} +$$

$$+ \frac{\rho}{n} \left[\left\{ \tau(\rho) - \epsilon \right\} r^{\rho} (l_1 r)^{\alpha_1} \ldots (l_k r)^{\alpha_k} - n \log r \right],$$

or,

$$X > \frac{n}{r^{\rho} (l_1 n)^{\alpha_1} \ldots (l_k n)^{\alpha_k}} \exp \left[\frac{\rho}{n} \{\tau(\rho) - \epsilon\}\, r^{\rho} (l_1 r)^{\alpha_1} \ldots (l_k r)^{\alpha_k} \right]$$

$$\geqslant \frac{n}{r^{\rho} (l_1 n)^{\alpha_1} \ldots (l_k n)^{\alpha_k}}\, e\, \frac{\rho}{n} \{\tau(\rho) - \epsilon\}\, r^{\rho} (l_1 r)^{\alpha_1} \ldots (l_k r)^{\alpha_k},$$

since $e^x \geqslant ex$ for all x. Hence for $r = r_p$, $n = n_p$

$$X > e\rho \{\tau(\rho) - \epsilon\} \left\{ \frac{l_1 r}{l_1 n} \right\}^{\alpha_1} \ldots \left\{ \frac{l_k r}{l_k n} \right\}^{\alpha_k}$$

$$= e\rho \{\tau(\rho) - \epsilon\} \left\{ \frac{l_1 r}{l_1 \nu(r)} \right\}^{\alpha_1} \ldots \left\{ \frac{l_k r}{l_k \nu(r)} \right\}^{\alpha_k}, \tag{3}$$

where $\nu(r)$ denotes the rank of the maximum term for $|z| = r$.

Now $\limsup_{r \to \infty} \dfrac{\log \nu(r)}{\log r} = \rho$ so that $\dfrac{\rho \log r}{\log \nu(r)} > 1 - \epsilon'$ for all large r.

Also if $a > 1$,

$$\log \mu(r) = \log \mu(a) + \int_a^r \frac{\nu(r)}{r}\,dr$$

$$< \log \mu(a) + \nu(r) \log r.$$

Hence from (1), we have for $r = r_p$,

$$\nu(r) \log r > \{\tau(\rho) - \epsilon\}\, r^\rho (l_1 r)^{\alpha_1} \dots (l_k r)^{\alpha_k} - O(1),$$

so that

$$\log \nu(r) + \log\log r > \log\{\tau(\rho) - \epsilon\} + \rho \log r + \alpha_1 l_2 r +$$
$$+ \dots + \alpha_k l_{k+1} r + o(1),$$

or,

$$\frac{\rho \log r}{\log \nu(r)} < 1 + o(1)$$

for sufficiently large $r = r_p$. Hence from (3) we have for $r = r_p$,

$$X > \frac{e\rho}{\rho^{\alpha_1}} \{\tau(\rho) - \epsilon\} \left\{\frac{\rho\, l_1 r}{l_1 \nu(r)}\right\}^{\alpha_1} \dots \left\{\frac{l_k r}{l_k \nu(r)}\right\}^{\alpha_k}$$

$$\sim \frac{e}{\rho^{\alpha_1 - 1}} \{\tau(\rho) - \epsilon\}.$$

Hence

$$\limsup_{n\to\infty} \frac{n}{(l_1 n)^{\alpha_1} \dots (l_k n)^{\alpha_k}} |a_n|^{\rho/n} \geqslant \frac{e}{\rho^{\alpha_1 - 1}} \tau(\rho).$$

If $\tau(\rho) = \infty$ the argument shows that $\vartheta = \infty$.

(ii) Suppose first that $0 < \odot < \infty$. Then

$$\frac{n}{(l_1 n)^{\alpha_1} \dots (l_k n)^{\alpha_k}} |a_n|^{\rho/n} > (\odot - \epsilon) \frac{e}{\rho^{\alpha_1 - 1}}$$

for $n > n_0$. By Cauchy's inequality $M(r) \geqslant |a_n|\, r^n$, and so, for $n > n_0$,

$$\frac{\log M(r)}{r^\rho (l_1 r)^{\alpha_1} \dots (l_k r)^{\alpha_k}} \geqslant \frac{\log |a_n| + n \log r}{r^\rho (l_1 r)^{\alpha_1} \dots (l_k r)^{\alpha_k}}$$

$$> \frac{n \log r + \frac{n}{\rho} \log \left\{ (\odot - \epsilon)\, e/\rho^{\alpha_1 - 1} \right\} - \frac{n}{\rho} \log \left\{ \frac{n}{(l_1 n)^{\alpha_1} \dots (l_k n)^{\alpha_k}} \right\}}{r^\rho (l_1 r)^{\alpha_1} \dots (l_k r)^{\alpha_k}}.$$

Let $A = \rho^{\alpha_1 - 1}/\odot$ and

$$\left\{ \frac{nA}{(l_1 n)^{\alpha_1} \dots (l_k n)^{\alpha_k}} \right\}^{1/\rho} \leqslant r < \left\{ \frac{(n+1)A}{\{l_1(n+1)\}^{\alpha_1} \dots \{l_k(n+1)\}^{\alpha_k}} \right\}^{1/\rho}$$

Then

$$\log r = \frac{1}{\rho} \log \left\{ \frac{n}{(l_1 n)^{\alpha_1} \dots (l_n n)^{\alpha_k}} \right\} + O\left(\frac{1}{n}\right),$$

and

$$\frac{\log M(r)}{r^\rho (l_1 r)^{\alpha_1} \dots (l_k r)^{\alpha_k}} >$$

$$\frac{\left\{l_1(n+1)\right\}^{\alpha_1} \dots \left\{l_k(n+1)\right\}^{\alpha_k} \left[\frac{n}{\rho} \log A + \log\left\{(\odot - \epsilon)e/\rho^{\alpha_1 - 1}\right\} + O(1) \right]}{(n+1)A \left[\frac{n}{\rho} \log \left\{ \frac{n}{(l_1 n)^{\alpha_1} \dots (l_k n)^{\alpha_k}} \right\} + O\left(\frac{1}{n}\right) \right]^{\alpha_1} \dots \left[l_k n + O(l_k n) \right]^{\alpha_k}}$$

$$\sim \frac{\rho^{\alpha_1 - 1}}{A} \log \left\{ (\odot - \epsilon) e A / \rho^{\alpha_1 - 1} \right\}$$

$$= \odot \log \left\{ (\odot - \epsilon) e / \odot \right\}.$$

Hence $r(\rho) \geqslant \odot$, which obviously holds when $\odot = 0$. If $\odot = \infty$ the above argument shows that $\tau(\rho) = \infty$.

(iii) If $\mu(r)$ denotes the maximum term of the series for $|z| = r$, then

$$\mu(r) = |a_n| r^n \text{ for } |a_{n-1}/a_n| \leqslant r < |a_n/a_{n+1}|.$$

Suppose first $\tau(\rho) < \infty$. Then

$$\log \mu(r) = \log |a_n| + n \log r > \{\tau(\rho) - \epsilon\} r^\rho (l_1 r)^{\alpha_1} \dots (l_k r)^{\alpha_k}, \quad (4)$$

for all $r > R_0$ and for all n such that $|a_{n-1}/a_n| \leqslant r < |a_n/a_{n+1}|$. If X is as defined in (3), then for $n > n_1$,

$$\log X = \log \left\{ \frac{n}{(l_1 n)^{\alpha_1} \dots (l_k n)^{\alpha_k}} \right\} + \frac{\rho}{n} \log |a_n|$$

$$> \log \left\{ \frac{n}{(l_1 n)^{\alpha_1} \dots (l_k n)^{\alpha_k}} \right\} +$$

$$+ \frac{\rho}{n} \left[\{\tau(\rho) - \epsilon\} r^\rho (l_1 r)^{\alpha_1} \dots (l_k r)^{\alpha_k} - n \log r \right],$$

or,

$$X > \frac{n}{r^{\rho}(l_1 n)^{\alpha_1}\dots(l_k n)^{\alpha_k}} \exp\left[\frac{\{\tau(\rho)-\epsilon\}\, r^{\rho}(l_1 r)^{\alpha_1}\dots(l_k r)^{\alpha_k}}{n}\right]$$

$$> \frac{n}{r^{\rho}(l_1 n)^{\alpha_1}\dots(l_k n)^{\alpha_k}}\, e\, \frac{\{\tau(\rho)-\epsilon\}\, r^{\rho}(l_1 r)^{\alpha_1}\dots(l_k r)^{\alpha_k}}{n}$$

$$= \frac{e\rho}{\rho^{\alpha_1}}\{\tau(\rho)-\epsilon\}\left\{\frac{\rho\, l_1 r}{l_1 \nu(r)}\right\}^{\alpha_1}\dots\left\{\frac{l_k r}{l_k \nu(r)}\right\}^{\alpha_k}$$

$$\sim \frac{e}{\rho^{\alpha_1-1}}\{\tau(\rho)-\epsilon\},$$

since $f(z)$ is of order $\rho(0<\rho<\infty)$ and (4) holds for all $r > R_0$. Hence

$$\liminf_{n\to\infty} \frac{n}{(l_1 n)^{\alpha_1}\dots(l_k n)^{\alpha_k}}\, |a_n|^{\rho/n} \geqslant \frac{e}{\rho^{\alpha_1-1}}\,\tau(\rho).$$

If $\tau(\rho)=\infty$ the above argument shows that $\odot = \infty$.

3. We consider now entire functions of zero order for which $M(r)$ is not very small. Let

$$\limsup_{r\to\infty} \frac{\log\log M(r)}{\log\log r} = \alpha_1. \tag{5}$$

THEOREM 2. *If* $f(z)=\sum_0^\infty a_n z^n$ *be of zero order and such that* $1<\alpha_1<\infty$, *then*

(i) $\tau(0)=u$,

(ii) $\tau(0)\geqslant v$,

(iii) *if further* $|a_n/a_{n+1}|$ *is a steadily increasing function of* n *for all large* n *then* $\tau(0)=u$.

PROOF. Suppose first that $0<u<\infty$ and let n_0 be so large that $l_k n_0 > 0$. Then

$$\log|a_n| > -(\alpha_1-1)\left\{\frac{(\alpha_1-1)^{\alpha_2}}{\alpha_1^{\alpha_1}}\, \frac{n^{\alpha_1}}{(l_1 n)^{\alpha_2}\dots(l_{k-1}n)^{\alpha_k}}\, \frac{1}{u-\epsilon}\right\}^{1/(\alpha_1-1)}$$

for a sequence of values of n tending to infinity, say, $n_1, n_2, \ldots, n_p, \ldots$ $(n_1 > n_0)$. By Cauchy's inequality $M(r) \geqslant |a_n|\, r^n$, and so for $n = n_p$,

$$\frac{\log M(r)}{(l_1 r)^{\alpha_1} \ldots (l_k r)^{\alpha_k}} \geqslant \frac{\log |a_n| + n \log r}{(l_1 r)^{\alpha_1} \ldots (l_k r)^{\alpha_k}}$$

$$> \frac{-(\alpha_1 - 1)\left\{\dfrac{(\alpha_1 - 1)^{\alpha_2}}{\alpha_1^{\alpha_1}} \cdot \dfrac{n^{\alpha_1}}{(l_1 n)^{\alpha_2} \ldots (l_{k-1} n)^{\alpha_k}} \cdot \dfrac{1}{u - \epsilon}\right\}^{1/(\alpha_1 - 1)} + n \log r}{(l_1 r)^{\alpha_1} \ldots (l_{k-1} r)^{\alpha_k}}$$

$$= \Bigg[- \left\{\frac{(\alpha_1 - 1)^{\alpha_2}}{\alpha_1} \cdot \frac{n^{\alpha_1}}{(l_1 n)^{\alpha_2} \ldots (l_k n)^{\alpha_k}} \cdot \frac{1}{u - \epsilon}\right\}^{1/(\alpha_1 - 1)} +$$

$$+ \left\{\frac{(\alpha_1 - 1)^{\alpha_2}}{\alpha_1^{\alpha_1}} \frac{n^{\alpha_1}}{(l_1 n)^{\alpha_2} \ldots (l_{k-1} n)^{\alpha_k}} \frac{1}{u - \epsilon}\right\}^{1/(\alpha_1 - 1)} + n \log r \Bigg] \times$$

$$\times \{(l_1 r)^{\alpha_1} \ldots (l_k r)^{\alpha_k}\}^{-1}.$$

Let

$$\log r = \log r_{n_p} = \left\{\frac{(\alpha_1 - 1)^{\alpha_2}}{\alpha_1} \cdot \frac{n_p}{(l_1 n_p)^{\alpha_1} \ldots (l_{k-1} n_p)^{\alpha_k}} \cdot \frac{1}{u - \epsilon}\right\}^{1/(\alpha_1 - 1)},$$

$$(p = 1, 2, \ldots).$$

Then

$$l_2 r_{n_p} = \frac{1}{\alpha_1 - 1} l_1 n_p + o(l_1 n_p); \quad l_3 r_{n_p} = l_2 n_p + o(l_2 n_p); \text{ etc.}$$

and so for $n = n_p$, $r = r_{n_p}$,

$$\frac{\log M(r)}{(l_1 r)^{\alpha_1} \ldots (l_k r)^{\alpha}}$$

$$> \frac{\left\{\dfrac{(\alpha_1 - 1)^{\alpha_2}}{\alpha_1^{\alpha_1}} \dfrac{n^{\alpha_1}}{(l_1 n)^{\alpha_2} \ldots (l_{k-1} n)^{\alpha_k}} \dfrac{1}{u - \epsilon}\right\}^{1/(\alpha_1 - 1)}}{\left\{\dfrac{(\alpha_1 - 1)^{\alpha_2}}{\alpha_1} \dfrac{n}{(l_1 n)^{\alpha_2} \ldots (l_{k-1} n)^{\alpha_k}} \dfrac{1}{u - \epsilon}\right\}^{\alpha_1/(\alpha_1 - 1)} \left\{\dfrac{1}{\alpha_1 - 1} l_1 n + o(l_1 n)\right\}^{\alpha_2} \ldots \left\{l_{k-1} n + o(l_{k-1} n)\right\}^{\alpha_k}}$$

$$\sim u - \epsilon.$$

Hence $\tau(0) \geqslant u$, which obviously holds when $u = 0$. If $u = \infty$, the above argument shows that $\tau(0) = \infty$.

To prove that $u \geqslant \tau(0)$ we consider first the case $\tau(0) < \infty$. For functions of zero order $\log M(r) \sim \log \mu(r)$ and therefore for a sequence of values of r, say, $r_1, r_2, \ldots, r_p, \ldots$

$$\log \mu(r) > \{\tau(0)-\epsilon\}\,(l_1 r)^{\alpha_1}\dots(l_k r)^{\alpha_k}. \tag{6}$$

Let $\mu(r_p) = |a_{n_p}|\, r_p^{\,n_p}$, then for $r = r_p$, $n = n_p$ we have

$$n \log r + \log |a_n| > \{\tau(0)-\epsilon\}\,(l_1 r)^{\alpha_1}\dots(l_k r)^{\alpha_k},$$

and hence for $p > p_0$,

$$\frac{\{\tau(0)-\epsilon\}\,(l_1 r)^{\alpha_1-1}(l_2 r)^{\alpha_2}\dots(l_k r)^{\alpha_k}}{\nu(r)} < 1.$$

If $Y = \dfrac{n^{\alpha_1}}{(l_1 n)^{\alpha_2}\dots(l_{k-1} n)^{\alpha_k}}\left(\dfrac{\alpha_1-1}{\log(1/|a_n|)}\right)^{\alpha_1-1}$, then for $r = r_p$, $n = n_p$,

$$\begin{aligned}
Y &> \frac{n^{\alpha_1}}{(l_1 n)^{\alpha_2}\dots(l_{k-1} n)^{\alpha}}\left[\frac{\alpha_1-1}{n\log r-\{\tau(0)-\epsilon\}\,(l_1 r)^{\alpha_1}\dots(l_k r)^{\alpha_k}}\right]^{\alpha_1-1}\\
&= \frac{n^{\alpha_1}}{(l_1 n)^{\alpha_2}\dots(l_{k-1} n)^{\alpha_k}}\cdot\frac{(\alpha_1-1)^{\alpha_1-1}}{(n\log r)^{\alpha_1-1}}\times\\
&\qquad\times\left[1-\frac{\{\tau(0)-\epsilon\}\,(l_1 r)^{\alpha_1-1}(l_2 r)^{\alpha_2}\dots(l_k r)^{\alpha_k}}{n}\right]^{-(\alpha_1-1)}\\
&\geqslant \frac{n^{\alpha_1}}{(l_1 n)^{\alpha_2}\dots(l_{k-1} n)^{\alpha_k}}\cdot\frac{(\alpha_1-1)^{\alpha_1-1}}{(n\log r)^{\alpha_1-1}}\cdot\frac{\alpha_1^{\alpha_1}}{(\alpha_1-1)^{\alpha_1-1}}\times\\
&\qquad\times\frac{\{\tau(0)-\epsilon\}\,(l_1 r)^{\alpha_1-1}(l_2 r)^{\alpha_2}\dots(l_k r)^{\alpha_k}}{n},
\end{aligned}$$

since $(1-x)^{-(\alpha-1)} \geqslant \dfrac{\alpha^\alpha x}{(\alpha-1)^{\alpha-1}}$ for all $x \leqslant 1$ and $\alpha > 1$. Hence

$$\begin{aligned}
Y &> \alpha_1^{\alpha_1}\{\tau(0)-\epsilon\}\left\{\frac{l_2 r}{l_1 n}\right\}^{\alpha_2}\dots\left\{\frac{l_k r}{l_{k-1} n}\right\}^{\alpha_k}\\
&= \alpha_1^{\alpha_1}\{\tau(0)-\epsilon\}\left\{\frac{l_2 r}{l_1 \nu(r)}\right\}^{\alpha_2}\dots\left\{\frac{l_k r}{l_{k-1} \nu(r)}\right\}^{\alpha_k}.
\end{aligned} \tag{7}$$

If (5) holds, we have [4, (2.6)]

$$\limsup_{r\to\infty}\frac{\log \nu(r)}{\log\log r} = \alpha_1 - 1,$$

so that

$$\frac{(\alpha_1-1)\log\log r}{\log \nu(r)} > 1-\epsilon'$$

for all large r. Also from (6) we have for $r = r_p$,

$$\nu(r)\log r > \{\tau(0)-\epsilon\}\,(l_1 r)^{\alpha_1}\dots(l_k r)^{\alpha_k} + O(1)$$

so that

$$\frac{(\alpha_1 - 1)\, l_2\, r}{l_1\, \nu(r)} > 1 + o(1)$$

for sufficiently large $r = r_p$. Hence from (7) we have for $r = r_p$,

$$Y > \frac{\alpha_1^{\alpha_1}}{(\alpha_1 - 1)^{\alpha_2}} \{\tau(0) - \epsilon\} \left\{ \frac{(\alpha_1 - 1)\, l_2\, r}{l_1\, \nu(r)} \right\}^{\alpha_2} \cdots \left\{ \frac{l_k\, r}{l_{k-1}\, \nu(r)} \right\}^{\alpha_k}$$

$$\sim \frac{\alpha_1^{\alpha_1}}{(\alpha_1 - 1)^{\alpha_2}} \{\tau(0) - \epsilon\}.$$

Hence

$$\limsup_{n \to \infty} \frac{n^{\alpha_1}}{(l_1\, n)^{\alpha_2} \ldots (l_{k-1}\, n)^{\alpha_k}} \left(\frac{\alpha_1 - 1}{\log (1/|a_n|)} \right)^{\alpha_1 - 1} \geqslant \frac{\alpha_1^{\alpha_1}}{(\alpha_1 - 1)^{\alpha_2}}\, \tau(0).$$

If $\tau(0) = \infty$ the argument shows that $u = \infty$. This completes the proof of (i).

We omit the proofs of (ii) and (iii).

In the end I must thank Dr. S. M. Shah for his helpful guidance.

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HOMOGENEOUS GAMES

By J. R. ISBELL*

The games considered in this paper are *simple* games of many players. In ordinary English they would be called games. One simply specifies that certain coalitions of players are *winning sets,* subject to the requirements that two disjoint sets cannot both be winning and that any set which contains a winning set is a winning set. For example, the stockholders of a corporation form a simple game in which a set is winning if it has more than half of the total number of shares of stock. This paper concerns games which are fair, or *homogeneous*; that is, given any two players, A, B, it is possible to permute the players so as to take all winning sets into winning sets and A into B.

Three questions are taken up, but only one is completely settled. A simple game is called *strong* if it is impossible to divide the players into two sets neither of which wins; that is, there are no ties. It is clear that there is no strong simple homogeneous game of two players. One might think that there is no such game having an even number of players, but one was discovered in the complete enumeration of six-player games [1]. It turns out that

THEOREM 1. *If n is a natural number, half of whose prime factors, or more, are odd, then there exists a strong simple homogeneous n-player game. If n is a power of 2, or if $n = 12$, there is no such game.*

The case $n = 12$ of Theorem 1 is established by means of Miller's enumeration of all transitive groups on twelve letters [2], and this method clearly will not help towards a complete solution of the problem. The first case not covered by the theorem is $n = 20$.

THEOREM 2. *If some winning set in a simple homogeneous n-player game contains only k players, then $n \leqslant k^2 - 1$.*

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The trouble with Theorem 2 is that it is not known whether $k^2 - 1$ may be replaced by $k^2 - k + 1$. There are infinitely many games for which the latter value is attained, based on the finite projective planes. One identifies the players with the points of the plane and defines a winning set as a set which contains a line. These games have been studied (in the case of Desarguesian planes) by Richardson [4], who shows that none of them is strong except the one corresponding to the plane having three points on each line. Each of these games has the same number of *minimal* winning sets as of players. Now it is known [1] that a strong simple n-player game has at least n minimal winning sets. It turns out that

THEOREM 3. *There are exactly three strong simple homogeneous games which have the same number of minimal winning sets as of players, one each for* $n = 1, 3, 7$. *In particular, every non-Desarguesian finite projective plane contains a set of points which meets every line but includes no line.*

We proceed to the proof of Theorem 1. If n is odd then there is a strong simple homogeneous n-player game in which a set wins if and only if it contains more than half the total number of players.

LEMMA 1. *If n is even then there exists a strong simple homogeneous n-player game if and only if there exists a transitive group of permutations on n letters which does not transform any set (of half the total number of letters) to its complement.*

PROOF. Given such a game, the group of all permutations which take all winning sets into winning sets clearly has the specified property. Conversely, given such a permutation group G on $n=2m$ letters, consider the sets S of m letters. Let $G(S)$ be the family of all sets T such that some permutation in G takes S into T. The various families $G(S)$ are disjoint; and if T is the complement of S, then $G(T)$ consists precisely of the complements of the members of $G(S)$. Since a set of m letters meets every other set of m letters except its complement, we may choose a family F of m-element sets which is maximal with respect to the property that no two of its members are disjoint and is composed of families $G(S)$. Then

F is transformed into itself by every permutation in G. Define a winning set as a set of m or more letters which meets every member of F. There are not two disjoint winning sets, for both would have to consist of m letters and then one of the two sets would be a member of F. Thus one has a simple game, and for the same reason, the game is strong. Every permutation in G transforms winning sets into winning sets, and the proof is complete.

LEMMA 2. *For every natural number n which is twice an odd number greater than* 1, *there is a strong simple homogeneous n-player game.*

PROOF. Number the $n = 2m$ players as follows : $1, 1', 2, 2',\ldots, m, m'$. The group G is generated as follows. For each subset A of the numbers $1,\ldots, m$ which has an even number of elements, G contains the permutation which exchanges i with i' for each i in A and leaves the other players fixed. Observe that these permutations form a commutative group H. G is generated by H and the one permutation π which takes each i to $i+1$, i' to $(i+1)'$, for $i = 1,\ldots m-1$, and m to 1, m' to $1'$. Evidently G is a transitive group. If S is a set of m players, the number of numbers i in $1,\ldots, m$ such that S contains exactly one of the pair i, i' is odd, and therefore no permutation in G takes S to its complement. Lemma 1 completes the proof.

A standard device for building up simple games is the *committee composition* [1], formed in the same way a number of committees might send delegates to form a higher committee. If G is an n-player game and $G_1,\ldots,G_n$ are simple games with sets of players $A_1,\ldots,A_n$ (disjoint or not), the committee composition $G(G_1,\ldots,G_n)$ is played by the union of $A_1,\ldots,A_n$. A set S is a winning set if and only if the set of all i, such that $S \cap A_i$ is a winning set in G_i, forms a winning set in G. If G and all G_i are strong, so is $G(G_1,\ldots,G_n)$. If G is homogeneous, all G_i are copies of one homogeneous game H, and the sets A_i are disjoint, then clearly $G(G_1,\ldots,G_n)$ is homogeneous. In particular, the set of all n for which strong simple homogeneous n-player games exist is closed under multiplication; and this completes the proof of the positive assertions of Theorem 1.

If n is a power of 2, then the binomial coefficient $\binom{2n}{n}$ is divisible by 2 but not by 4, as is clear from writing it as a product. This is the number of n-element subsets of a set of $2n$ elements. Thus a homogeneous $2n$-player strong simple game would have to have an odd number of n-element winning sets. But then it is impossible for each player to be a member of the same number of n-element winning sets, contrary to homogeneity.

Finally, for $n = 12$ one must verify that none of the 295 groups in Miller's list [2] satisfies the criterion of Lemma 1. Though the verification cannot be reproduced here, it seems well to give hints for the two groups which took most time for the author. For the primitive group of order 7920, Miller gives four generators, in the order w, x, y, z; the product $wxzwx$ (operating on the right) violates Lemma 1. For the quintuply transitive group, one may easily prove that quintuple transitivity is impossible for a 12-player game.

This accounts for Theorem 1. Theorem 2 may be proved by either of two computations. In the first place, a winning set S in a homogeneous simple game on the set N of n players has the property that there is a transitive group G of permutations of N such that S meets every set into which it is transformed by G. Let $x_1, \ldots, x_k$ be the elements of S. For each g in G there is at least one solution of $x_i g = x_j$ $(1 \leqslant i, j \leqslant k)$. Now the set of all permutations in G which leave one element fixed is a subgroup of index n; the number of permutations in G is a multiple of n, hn; and for each i and j the number of solutions of $x_i g = x_j$ is exactly h. There are k^2 ordered pairs (x_i, x_j), and altogether hk^2 triples (i, j, g) solving $x_i g = x_j$. This is enough to provide a solution for each g, and more than one solution for the identity permutation. Therefore $hk^2 > hn$ and $k^2 > n$; since these are whole numbers, $n \leqslant k^2 - 1$.

Alternatively, let us consider the k-element winning sets, or *k-sets* for short. Each player is in the same number p of k-sets. Choose a k-set S. Every other k-set meets S, and since each player in S is in exactly $p - 1$ other k-sets, there are at most $1 + k(p - 1)$

k-sets in all. The number of occurrences of players in k-sets is at most $k(1+k(p-1))$. Since each player occurs p times, there are at most $k^2-(k^2-k)/p$ players; that is, $n<k^2$.

If k is the number of points on each line in some finite projective plane, the number of points in the plane is $n=k^2-k+1$. There is an n-player homogeneous simple game whose players are the points, in which a set wins if and only if it contains a line. One can even make the game strong, and still homogeneous, by permitting a set to win if it contains more than half the total number of points and meets every line. The first case in which k^2-k+1 is a better bound than k^2-1 is $k=3$, $k^2-1=8$; and by Theorem 1 there is no 8-player strong simple homogeneous game. One can easily show by special arguments that no 8-players homogeneous simple game has a winning set consisting of only 3 players. The next case, $k=4$, $n=14$ or 15, is unsettled.

Proceeding to Theorem 3, we consider n-player homogeneous strong simple games with n minimal winning sets. If the smallest winning sets have k members, we can improve Theorem 2 by noting the inequality $n\leqslant k^2-(k^2-k)/p$ established in the second proof. In the present case there are at most n winning sets of k players, and thus each player is in at most k of them. Then $p\leqslant k$ and $n\leqslant k^2-k+1$.

Lemma 3. *If a strong simple n-player game has no winning sets of fewer than k players and only n minimal winning sets, then* $n\geqslant 2^{k-1}$.

Proof. Suppose first that all minimal winning sets are of size k and that n is an even number $2m$. We refer to the minimal winning sets as k-sets, and to any set of size m as an m-set. There are only n k-sets, but there are $\binom{2m}{m}$ m-sets, and each m-set either contains a k-set or is disjoint from one. Each k-set is contained in $\binom{2m-k}{m}$ m-sets and is disjoint from the same number of m-sets. Therefore it must be true that $2.2m\binom{2m-k}{m}\geqslant\binom{2m}{m}$, and this is easily

seen to imply $n = 2m \geqslant 2^{k-1}$. In case some of the minimal winning sets contain more than k players, these are contained in even fewer m-sets and disjoint from fewer. Hence the lemma is established for even n. For the case $n = 2m + 1$, a similar argument on m-sets yields the same result.

Under the hypothesis of Theorem 3 this gives $k^2 - k + 1 \geqslant n \geqslant 2^{k-1}$, $k \leqslant 5$, $n \leqslant 21$. The cases $k = 4$ or 5 ($8 \leqslant n \leqslant 13$, $16 \leqslant n \leqslant 21$, respectively) are impossible, as is shown by repeating the argument of Lemma 3 but using exact values for the binomial coefficients instead of estimates. Now there are complete lists of strong simple games for $n \leqslant 5$ [3] and for $n = 6$ [1], and they include no games satisfying Theorem 3 except the unique 1-player and 3-player games. The exceptional 7-player game is discussed in [3]; it remains to show that it is unique. But in this case, by the second proof of Theorem 2, there can be seven minimal winning sets only if the intersection of any two minimal winning sets consists of just one player. Then the players and minimal winning sets satisfy the axioms for the points and lines of a projective plane (with a little further argument); and it is well known, and easily proved, that the seven-point projective plane is unique.

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EIGENVALUES AND CANONICAL FORMS OF MATRICES WITH REAL QUATERNION ELEMENTS

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1. Introduction. Matrices over real quaternions have been studied by some authors by using a certain isomorphism between quaternion matrices and complex matrices. In this paper we develop the theory of eigenvalues and canonical forms for quaternion matrices from first principles without using any such isomorphism. Actually this approach seems to be more simple. Using the relation of similarity in the set of real quaternions we have the general notion of an eigenvalue class of a quaternion matrix. By means of the minimum real polynomial of a quaternion matrix we obtain complete information regarding the eigenvalue classes and prove that a quaternion matrix is similar to a diagonal matrix if and only if its minimum real polynomial is a product of distinct irreducible real factors. For any quaternion matrix we obtain the Jordan normal form in a straightforward way and use this to obtain a necessary and sufficient condition for similarity of quaternion matrices.

2. We identify the algebras of real numbers and complex numbers as sub-algebras of the algebra of real quaternions. Every quaternion satisfies an irreducible real (monic) polynomial. Two quaternions a and b are said to be *similar* if there is a non-zero quaternion c such that $c^{-1}ac = b$. This is an equivalence relation and by elementary algebra it follows that two quaternions are similar if and only if they satisfy the same irreducible real polynomial. Thus equivalence classes are characterized by irreducible real polynomials. If $\mathscr{C}$ is an equivalence class and $p(x)$ the corresponding irreducible real polynomial, we shall call $p(x)$ the irreducible real polynomial of the class $\mathscr{C}$. We note that $p(x)$ is linear or quadratic according as the class $\mathscr{C}$ is real or not. Every quaternion λ is similar to its conjugate $\bar{\lambda}$ and is also similar to a complex number.

From elementary considerations we obtain the following results regarding quaternions.

LEMMA 2.1. *Given a, b, c where a is not similar to b, we can find y such that $ay - yb = c$.*

LEMMA 2.2. *If λ is not a real number then the general solution of $\lambda y = y\bar{\lambda}$ is $y = \lambda z - z\bar{\lambda}$ where z is arbitrary.*

LEMMA 2.3. *If λ is not a real number then for every a, we can find b and c such that $a = b + \lambda c - c\lambda$ where $b\lambda = \lambda b$. In particular if λ is complex (but not real) then b is also complex.*

These results are useful in what follows.

The operations of addition, product and scalar multiplication for quaternion matrices are defined in the usual way, it being noted that if A is a matrix and λ a quaternion then in general $A\lambda \neq \lambda A$. However, if λ is real then necessarily $A\lambda = \lambda A$.

A square matrix A of order n is said to be *singular* if there is an n-rowed column vector $X \neq 0$ with $AX = 0$. Equivalently A is singular if there is an n-columned row vector $Y \neq 0$ with $YA = 0$. If A is not singular then it has a unique inverse A^{-1} such that $AA^{-1} = A^{-1}A =$ the unit matrix I.

A matrix A is said to be of *rank* r if r is the dimension of the right vector space (over quaternions) spanned by the columns of A. As in the complex case, r is also the dimension of the left vector space spanned by the rows of A. If A is of order n and rank r then $\rho = n-r$ is called the *nullity* of A. ρ is the dimension of the right vector space of column vectors X satisfying $AX = 0$ (and also the dimension of the left vector space of row vectors Y, satisfying $YA = 0$).

We shall denote by A^* the transpose of the conjugate of A. Then from $\overline{ab} = \bar{b}\bar{a}$ we obtain $(AB)^* = B^*A^*$.

A square matrix A is said to be *hermitian*, *unitary* or *normal* according as $A^* = A$, $A^* = A^{-1}$ or $AA^* = A^*A$. Triangle and

nilpotent matrices are defined as usual and by induction one can easily show that a triangle normal matrix is necessarily diagonal.

Two square matrices A, B of order n are said to be *similar* if there is a non-singular matrix S of order n such that $S^{-1}AS = B$. A is said to be *diagonable* if it is similar to a diagonal matrix. Just as every quaternion is similar to a complex number, we shall prove that every quaternion matrix is similar to a complex matrix.

3. Eigenvalues. Let A be a square matrix of order n. A quaternion λ is called an *eigenvalue* of A if there is a *non-zero* n-rowed column vector X such $AX = X\lambda$.

It follows from the definition that similar matrices have the same eigenvalues. If $AX = X\lambda$, $X \neq 0$ then for any $\rho \neq 0$ we have $A(X\rho) = (X\rho)\ \rho^{-1}\lambda\rho$ and $X\rho \neq 0$ so that $\rho^{-1}\lambda\rho$ is also an eigenvalue of A. This proves

THEOREM 3.1. *If λ is an eigenvalue of A so is every quaternion similar to λ.*

This means that either every element of a class $\mathscr{C}$ is an eigenvalue of A or no element of $\mathscr{C}$ is an eigenvalue of A. Thus we can speak of the *eigenvalue classes* of A. We have the following characterization of an eigenvalue class.

THEOREM 3.2. *$\mathscr{C}$ is an eigenvalue class of A if and only if $p(A)$ is singular, where $p(x)$ is the irreducible real polynomial of the class $\mathscr{C}$.*

PROOF. The theorem is trival if $p(x)$ is linear. Let

$$p(x) = x^2 - tx + n.$$

Then if $\lambda \in \mathscr{C}$, $\bar{\lambda} \in \mathscr{C}$ and $\lambda + \bar{\lambda} = t$, $\lambda\bar{\lambda} = n$. nP. Suppose that $p(A) = A^2 - tA + nI$ is singular. Then there exists a column vector $X \neq 0$ such that $(A^2 - tA + nI)\ X = 0$, or since t and n are real, $A^2X - AXt + Xn = 0$. Thus

$$A(AX - X\lambda) - (AX - X\lambda)\ \bar{\lambda} = 0.$$

Now if $AX - X\lambda \neq 0$, then by the above, $\bar{\lambda}$ is an eigenvalue of A. On the other hand if $AX - X\lambda = 0$ then since $X \neq 0$, λ is an

eigenvalue of A. Thus in any case $\mathscr{C}$ is an eigenvalue class of A. The converse is trivial.

We can also define an eigenvalue λ by the condition $YA = \lambda Y$ for some non-zero row vector Y. Then following this definition we prove Theorems 3.1 and 3.2 and thus establish the equivalence of the two definitions.

Now to prove the existence of eigenvalue classes we proceed as follows. It is clear that any square matrix A satisfies a polynomial equation with real coefficients. Let $\phi(x)$ be the *monic* polynomial of *least* degree with *real* coefficients such that $\phi(A) = O$. Then $\phi(x)$ is uniquely determined for A and we call $\phi(x)$ the *minimum real polynomial* of A. Let $\phi(x) = \prod_{i=1}^{m} [p_i(x)]^{s_i}$ where $p_1(x)$, $p_2(x), \ldots,$ $p_m(x)$ are the distinct irreducible real factors of $\phi(x)$. Then $O = \phi(A) = \prod_{i=1}^{m} [p_i(A)]^{s_i}$ and by the minimal nature of $\phi(x)$ it follows that each of $p_i(A)$, $i = 1, 2, \ldots, m$, is singular.

If $\mathscr{C}_i$ is the class corresponding to $p_i(x)$ then by Theorem 3.2, $\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_m$ are eigenvalue classes of A. We now show that these are just all the distinct eigenvalue classes of A.

For let $\mathscr{C}$ be any class and $p(x)$ its irreducible real polynomial. If $p(x)$ is not a factor of $\phi(x)$ then $p(x)$ is coprime to $\phi(x)$ and we can find *real* polynomials $g(x)$, $h(x)$ such that

$$p(x)g(x) + \phi(x)h(x) = 1.$$

Then

$$p(A)g(A) + \phi(A)h(A) = I.$$

Since $\phi(A) = O$, we see that $p(A)$ is non-singular so that by Theorem 3.2, $\mathscr{C}$ cannot be an eigenvalue class of A. Thus we have

THEOREM 3.3. *Every square matrix A has a finite number of distinct eigenvalue classes. These correspond to the distinct irreducible real factors of the minimum real polynomial $\phi(x)$ of A.*

The eigenvalues can be computed by using the regular representation of quaternion matrices by real matrices. If $1, i, j, k$ is

the usual base for the quaternion algebra, then any square matrix A of order n can be uniquely expressed as $A = A_0 + A_1 i + A_2 j + A_3 k$ where A_0, A_1, A_2, A_3 are real square matrices of order n. In the regular representation we associate to A, the real matrix

$$\tilde{A} = \begin{bmatrix} A_0 & -A_1 & -A_2 & -A_3 \\ A_1 & A_0 & -A_3 & A_2 \\ A_2 & A_3 & A_0 & -A_1 \\ A_3 & -A_2 & A_1 & A_0 \end{bmatrix}.$$

It is clear that $\phi(x)$, the minimum real polynomial of A, is also the minimum polynomial of $\tilde{A}$ so that *the eigenvalues of A are just all the quaternions which are similar to the complex eigenvalues of $\tilde{A}$.* The eigenvalues of $\tilde{A}$ are known from the characteristic polynomial of $\tilde{A}$. Thus the eigenvalues of A can be found out.

We make the following remarks.

Lee [4] defines an eigenvalue of A as a *complex number* λ such that $AX = X\lambda$ for some non-zero column vector X. Brenner [1] allows quaternion eigenvalues and proves their existence, but our method is simple and gives complete information regarding the eigenvalue classes. In a way this justifies the definition given by Lee [4] since according to our analysis, it is sufficient to consider only complex eigenvalues, all other eigenvalues being similar to these complex eigenvalues. The other definition of an eigenvalue, viz. $YA = \lambda Y$ for some non-zero row vector Y, does not seem to have been considered by other authors. It is interesting to note that the two definitions are equivalent.

Now we have many results analogous to the complex case. First we note

LEMMA 3.1. *The eigenvalues of* $\begin{pmatrix} A & C \\ O & B \end{pmatrix}$ *where A, B are square matrices, are just the eigenvalues of A and the eigenvalues of B.*

PROOF. Let $AX = X\lambda$, $YB = \mu Y$, $X \neq 0$, $Y \neq 0$. Then

$$\begin{pmatrix} A & C \\ O & B \end{pmatrix} \begin{pmatrix} X \\ 0 \end{pmatrix} = \begin{pmatrix} X \\ 0 \end{pmatrix} \lambda, \quad (0\ Y) \begin{pmatrix} A & C \\ O & B \end{pmatrix} = \mu(0,\ Y)$$

shows that the eigenvalues of A, B are also eigenvalues of $\begin{pmatrix} A & C \\ O & B \end{pmatrix}$.

Conversly, let

$$\begin{pmatrix} A & C \\ O & B \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \lambda,$$

where $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \neq 0$, then $AX_1 + CX_2 = X_1\lambda$ and $BX_2 = X_2\lambda$. If $X_2 \neq 0$, λ is an eigenvalue of B. If $X_2 = 0$ then $AX_1 = X_1\lambda$ and $X_1 \neq 0$ so that λ is an eigenvalue of A. From this we immediately have by induction the following

THEOREM 3.4. *The eigenvalues of a triangle matrix are just the diagonal elements and all quaternions similar to them.*

This implies that a *triangle matrix of order n has at most n distinct eigenvalue classes.*

If X is an n-rowed column vector satisfying $X^*X = 1$, then as in the complex case we can find a unitary matrix U of order n, with X as the first column. Using this and applying induction we obtain

THEOREM 3.5. *Any square matrix is unitary-similar to a triangle matrix, i.e. for a square matrix A, there exists a unitary matrix U such that $U^{-1}AU = T$ is a triangle matrix.*

COROLLARY i. *A square matrix of order n has at most n distinct eigenvalue classes.*

COROLLARY ii. *A is unitary-similar to a diagonal matrix if and only if A is normal.*

4. Reduction to canonical form. We now proceed to obtain a canonical form for a square matrix under similarity operation. Since every square matrix is similar to a triangle matrix it is sufficient to consider the case of a triangle matrix.

Let T be a triangle matrix and $\mathscr{C}$ an eigenvalue class of T. If r places in the diagonal of T are occupied by elements of $\mathscr{C}$, we shall call r the *multiplicity* of the eigenvalue class $\mathscr{C}$. If two triangle matrices are similar they have the same eigenvalue classes. We assert that their multiplicities are also the same. This is trivial in the complex case because of the notion of the characteristic polynomial. In the quaternionic case we prove the result as follows.

We require the following

LEMMA 4.1. *If S and T are two triangle matrices of orders m and n respectively without any common eigenvalue class then for any $m\times n$ matrix P, $SP = PT$ implies $P = 0$.*

The proof of this is straightforward.

Now let A and B be two triangle matrices of order n and $\mathscr{C}$ an eigenvalue class with multiplicities r in A and s in B where $r < s$. We show that A cannot be similar to B. Since any permutation among the diagonal elements of a triangle matrix gives us a triangle matrix similar to the original one, we may take A, B in the form

$$A = \begin{pmatrix} A_1 & A_2 \\ O & A_4 \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & B_2 \\ O & B_4 \end{pmatrix},$$

where A_1, B_1 are triangle matrices of order r with only one eigenvalue class $\mathscr{C}$. Since $r < s$, $\mathscr{C}$ is an eigenvalue class of B_4 but not of A_4 so that A_4 cannot be similar to B_4.

Let $AP = PB$, where P is of order n. Write $P = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix}$ where P_1 is of order r. Then

$$\begin{pmatrix} A_1 & A_2 \\ O & A_4 \end{pmatrix} \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix} = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix} \begin{pmatrix} B_1 & B_2 \\ O & B_4 \end{pmatrix}$$

gives in particular $A_4P_3 = P_3B_1$ and $A_4P_4 = P_4B_4 + P_3B_2$. Here B_1 is a triangle matrix of order r with only one eigenvalue class $\mathscr{C}$. A_4 is a triangle matrix of order $n-r$, for which $\mathscr{C}$ is not an eigenvalue class and P_3 is an $(n-r)\times r$ block. Hence by Lemma 4.1, $A_4P_3 = P_3B_1$ implies $P_3 = 0$. Then we have $A_4P_4 = P_4B_4$.

Since A_4 is not similar to B_4 this implies that P_4 is singular. Thus $P = \begin{pmatrix} P_1 & P_2 \\ O & P_4 \end{pmatrix}$ where P_4 is singular. Therefore P is singular. Hence A cannot be similar to B.

This enables us to define the multiplicities of the eigenvalue classes of any square matrix A in the following way. The multiplicity of an eigenvalue class $\mathscr{C}$ of A is the multiplicity of $\mathscr{C}$ in any triangle matrix T similar to A. We shall, later on, show how to compute these multiplicities.

As a first step towards the canonical form we prove

THEOREM 4.1. *Let A be a square matrix with distinct eigenvalue classes $\mathscr{C}_i$ of multiplicities r_i, $i = 1, 2, \ldots, m$. Then A is similar to a matrix of the form*

$$\text{diag}\,(T_1, T_2, \ldots, T_m),$$

where T_i is a triangle matrix of order r_i with only one eigenvalue class $\mathscr{C}_i$, $i = 1, 2, \ldots, m$.

PROOF. Without any loss of generality we may assume that A itself is a triangle matrix of the form $\begin{pmatrix} T_1 & C \\ O & B \end{pmatrix}$ where T_1 is a triangle matrix of order r_1 with only one eigenvalue class $\mathscr{C}_1$ and B is a triangle matrix whose distinct eigenvalue classes are $\mathscr{C}_p$ of multiplicities r_p, $p = 2, 3, \ldots, m$.

We apply induction on m, the number of distinct eigenvalue classes of A. For $m = 1$, the theorem is trivial. Assume the theorem proved for $m - 1$. Then B is similar to diag $(T_2, T_3, \ldots, T_m)$ where T_p is a triangle matrix of order r_p with only one eigenvalue class $\mathscr{C}_p$, $p = 2, 3, \ldots, m$. Thus A is similar to

$$A_1 = \begin{bmatrix} T_1 & C_2 & C_3 & \ldots & C_m \\ O & T_2 & O & \ldots & O \\ O & O & T_3 & \ldots & O \\ O & O & \ldots & \ldots & T_m \end{bmatrix}$$

Now to annihilate the blocks $C_2, C_3, \ldots C_m$ we use the following generalization of Lemma 2.1.

LEMMA 4.2. *Let T_1, be a triangle matrix of order r with only one eigenvalue class $\mathscr{C}_1$ and T_2 a triangle matrix of order s with only one eigenvalue class $\mathscr{C}_2$ where $\mathscr{C}_1$ is distinct from $\mathscr{C}_2$. Then for any $r \times s$ matrix C we can find an $r \times s$ matrix P such that $T_1 P - P T_2 + C = O$.*

Thus we have an $r_1 \times r_p$ matrix U_p such that

$$T_1 U_p - U_p T_p + C_p = O, \quad p = 2, 3, \ldots m.$$

Write

$$U = \begin{bmatrix} I_{r_1} & U_2 & U_3 & \ldots & U_m \\ O & I_{r_2} & O & \ldots & O \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ O & O & \ldots & \ldots & I_{r_m} \end{bmatrix}$$

where I_r is the unit matrix of order r. Then we have

$$U^{-1} A_1 U = \operatorname{diag}(T_1, T_2, \ldots, T_m).$$

Thus A is similar to diag $(T_1, T_2, \ldots, T_m)$ and the proof is completed by induction.

This enables us to obtain a necessary and sufficient condition for A to be similar to a diagonal matrix. First, if each $T_i, i = 1, 2, \ldots, m$, is diagonable then so is A. Conversely, if A is diagonable then by using Lemma 4.1, one can show that each T_i is diagonable.

Now let T be a triangle matrix with only one eigenvalue class $\mathscr{C}$ and $p(x)$ the irreducible real polynomial of $\mathscr{C}$. If T is diagonable then clearly $p(T) = O$. Conversely if $p(T) = O$ then we show that T is diagonable. This is trivial if $p(x)$ is linear.

Let $p(x) = x^2 - tx + n$, $\lambda \in \mathscr{C}$ and $T = \begin{pmatrix} \lambda & \xi \\ O & S \end{pmatrix}$ be of order r. Then S is a triangle matrix of order $r-1$ with only one eigenvalue class $\mathscr{C}$ and $p(T) = O$ implies $p(S) = O$. Let us assume the result proved for $r-1$, then S is diagonable and hence similar to λI. Thus T is similar to $T_1 = \begin{pmatrix} \lambda & \eta \\ O & \lambda I \end{pmatrix}$. Now $p(T) = O$ implies $p(T_1) = O$ which gives $\lambda\eta + \eta\lambda - t\eta = 0$ or $\lambda\eta = \eta\bar{\lambda}$, as $t = \lambda + \bar{\lambda}$. Since λ is

not real, by Lemma 2.2 we can write $\eta = \lambda\zeta - \zeta\lambda$, where ζ is a row vector. Call $U = \begin{pmatrix} 1 & -\zeta \\ O & I \end{pmatrix}$. Then $U^{-1}T_1U = \lambda I$. Thus T is diagonable and the proof is completed by induction. Thus a triangle matrix T with only one eigenvalue class $\mathscr{C}$ is diagonable if and only if $p(T) = O$ where $p(x)$ is the irreducible real polynomial of $\mathscr{C}$.

Now the minimum real polynomial of A is the product of the minimum real polynomials of $T_1, T_2, \ldots, T_m$ so that A is diagonable if and only if $\phi(x) = \prod_{i=1}^{m} p_i(x)$ is the minimum real polynomial of A. Thus we have

THEOREM 4.2. *A square matrix is similar to a diagonal matrix if and only if its minimum real polynomial is a product of distinct irreducible real factors.*

COROLLARY. *If $\tilde{A}$ corresponds to A in the regular representation, then A is diagonable if and only if $\tilde{A}$ is diagonable.*

Let $\phi(x) = \prod_{i=1}^{m} [p_i(x)]^{s_i}$ be the minimum real polynominl of A and A similar to diag $(T_1, T_2, \ldots, T_m)$ as in Theorem 4.1. Then $[p_i(x)]^{s_i}$ is the minimum real polynominal of T_i and for $i \neq j$, $p_i(T_j)$ is non-singular. Hence for ρ_i, $1 \leqslant \rho_i \leqslant s_i$ the nullity of $[p_i(A)]^{\rho_i}$ is the same as the nullity of $[p_i(T_i)]^{\rho_i}$. In particular, *the nullity of $[p_i(A)]^s$ is equal to r_i, the multiplicity of the eigenvalue class $\mathscr{C}_i$.*

We now consider non-diagonable matrices and obtain their Jordan normal form. We shall denote by $J_\lambda^{(r)}$ the Jordan matrix of order r:

$$J_\lambda^{(r)} = \begin{bmatrix} \lambda & 1 & 0 & \ldots & \ldots & 0 \\ 0 & \lambda & 1 & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \lambda & 1 \\ 0 & 0 & \ldots & \ldots & 0 & \lambda \end{bmatrix}.$$

It is clear that the minimum real polynomial of $J_{\lambda}^{(r)}$ is $[p(x)]^r$, where $p(x)$ is the irreducible real polynomial of λ. Also for $1 \leqslant t \leqslant r$, the rank of $[p(J_{\lambda}^{(r)})]^t$ is $r-t$ and therefore the nullity is t.

Let A be a triangle matrix of order n with only one eigenvalue class $\mathscr{C}$ and let the minimum real polynomial of A be $[p(x)]^m$ where $p(x)$ is the irreducible real polynomial of the class $\mathscr{C}$.

Let $\lambda \in \mathscr{C}$. Without any loss of generality we may assume that the diagonal elements of A are each equal to λ and that λ *is complex.* Thus $A = \lambda I + B$ where B is a triangle matrix whose diagonal elements are zero. Consequently B is nilpotent. Now the proof due to the author [2] applies equally to the quaternionic case and there is a non-singular matrix S such that

$$S^{-1}BS = \text{diag } (J_0^{(e_1)}, J_0^{(e_2)}, \ldots, J_0^{(e_p)}).$$

But unless λ is real, $S^{-1}\, \lambda I\, S$ is not necessarily equal to λI. If λ is *real* then $S^{-1}\, \lambda IS = \lambda I$ so that we have,

$$S^{-1}AS = \text{diag}\,(J_{\lambda}^{(e_1)}, J_{\lambda}^{(e_2)}, \ldots, J_{\lambda}^{(e_p)}).$$

When λ is not real we proceed as follows. We use the following generalization of Lemma 2.3.

LEMMA 4.3. *Given* $J = \text{diag } (J_{\lambda}^{(f_1)}, J_{\lambda}^{(f_2)}, \ldots, J_{\lambda}^{(f_q)})$ *where* λ *is not real, we can find for any row vector* ξ *(with the same number of columns as* J*) two other row vectors* η *and* ζ *such that*

$$\xi = \eta + \lambda\zeta - \zeta J, \textit{ where } \lambda\eta = \eta\,\lambda.$$

Further if λ *is complex* ($\neq$ *real) then* η *is a complex vector.*

To obtain the Jordan normal form of A we apply induction on nth order of A. We have $A = \begin{pmatrix} \lambda & \theta \\ O & C \end{pmatrix}$ where θ is a row vector and C is a triangle matrix of order $n-1$, with diagonal elements each equal to λ. Though A is non-diagonable, C may or may not be diagonable. In any case by induction hypothesis C is similar to

$$J = \text{diag } (J_{\lambda}^{(f_1)}, J_{\lambda}^{(f_2)}, \ldots, J_{\lambda}^{(f_q)}),$$

and thus A is similar to $A_1 = \begin{pmatrix} \lambda & \xi \\ O & J \end{pmatrix}$. By Lemma 4.3 we can

write $\xi = \eta + \lambda\zeta - \zeta J$, where η is a *complex row vector*, since λ is complex ($\neq$ real). Hence

$$\begin{pmatrix} \lambda & \xi \\ O & J \end{pmatrix} \begin{pmatrix} 1 & -\zeta \\ O & I \end{pmatrix} = \begin{pmatrix} 1 & -\zeta \\ O & I \end{pmatrix} \begin{pmatrix} \lambda & \eta \\ O & J \end{pmatrix}.$$

Thus A_1 is similar to $A_2 = \begin{pmatrix} \lambda & \eta \\ O & J \end{pmatrix}$ which is a complex matrix. Since A is non-diagonable A_2 is non-diagonable and by results already known for complex matrices, A_2 is similar to

$$\text{diag } (J_\lambda^{(e_1)}, (J_\lambda^{(e_2)}, \ldots J_\lambda^{(eq)})$$

and the proof is completed by induction.

Without any loss of generality we may take $1 \leqslant e_1 \leqslant e_2 \ldots \leqslant e_p$. Then the minimum real polynomial of A is $[p(x)]^{e_p}$ so that $e_p = m$. The other indices can be calculated by considering the ranks of $[p(A)]^r$ where $1 \leqslant r < m$.

If λ is similar to μ then $J_\lambda^{(r)}$ is similar to $J_\mu^{(r)}$ so that in the Jordan normal form of A, λ may be any eigenvalue of A, not necessarily complex. Combining this with Theorem 4.1, we obtain

THEOREM 4.3. *Let A be a square matrix with distinct eigenvalue classes $\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_m$ and the minimum real polynomial $\prod_{i=1}^{m} [p_i(x)]^{s_i}$ where $p_i(x)$ is the irreducible real polynomial of the class $\mathscr{C}_i$. Let $\lambda_i \in \mathscr{C}_i$. Then A is similar to* diag $(J_1, J_2, \ldots, J_m)$, *where*

$$J_i = \text{diag } (J_{\lambda_i}^{(e_{i_1})}, J_{\lambda_i}^{(e_{i_2})}, \ldots J_{\lambda i}^{(e_{it_i})}), \quad 1 \leqslant e_{i_1} \leqslant e_{i_2} \leqslant \ldots \leqslant e_{it_i} = s_i,$$

and

$$J_\lambda^{(r)} = \begin{vmatrix} \lambda & 1 & 0 & . & . & 0 \\ 0 & \lambda & 1 & 0 & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & \lambda & 1 \\ 0 & 0 & . & . & 0 & \lambda \end{vmatrix}$$

is of order r.

Wiegmann [5] has obtained the Jordan normal form by using a certain isomorphism between quaternion matrices and complex matrices.

The indices e_{ij}, $j = 1, 2, \ldots, t_i$; $i = 1, 2, \ldots, m$, can be uniquely determined by considering the ranks of $[p_i(A)]^r$, $r = 1, 2, \ldots, s_i$; $i = 1, 2, \ldots, m$.

This gives us the following condition for similarity of quaternion matrices.

THEOREM 4.4. *Let the square matrices A, B of order n have the same minimum real polynomial $\prod_{i=1}^{m} [p_i(x)]^{s_i}$. Then A is similar to B if and only if the rank of $[p_i(A)]^r$ is the same as the rank of $[p_i(B)]^r$ for every $r = 1, 2, \ldots, s_i$; $i = 1, 2, \ldots, m$.*

A more general result concerning similarity of matrices whose elements belong to a division algebra has been obtained by Ingham and Wolf [3] by using a different method.

The multiplicities of the eigenvalue classes of A can be computed from the characteristic polynomial of $\tilde{A}$ for considering the Jordan normal form of A in which the diagonal elements are complex. It is easy to see that the characteristic polynomial of $\tilde{A}$ is $\prod_{i=1}^{m} [p_i(x)]^{t_i}$, where $t_i = 4r_i$ or $2r_i$ according as $p_i(x)$ is linear or quadratic and r_i is the multiplicity of the eigenvalue class $\mathscr{C}_i$ corresponding to the irreducible real factor $p_i(x)$.

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ON DIFFERENTIATION UNDER THE INTEGRAL SIGN

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THE existing proofs of the formula for differentiation under the integral sign for Riemann integrals, viz.

$$\frac{d}{dy}\int_{a(y)}^{b(y)} f(x,y)\,dx = \int_{a(y)}^{b(y)} f_y(x,y)\,dx + b'(y)\,f(b(y),y) - a'(y)\,f(a(y),y)$$

make use, tacitly or explicitly, of the continuity of $f(x,y)$ and $f_y(x,y)$ in an *open domain* containing the closed set of points : $\alpha \leqslant y \leqslant \beta$, and $a(y) \leqslant x \leqslant b(y)$. The appeal to such conditions becomes obvious in case $a(y)$ or $b(y)$ is oscillatory at the point under consideration. The following theorem provides a proof of the formula free from this defect in the general case of Lebesgue integrals.

THEOREM. *Suppose that*

(i) *$a(y)$ and $b(y)$ are each differentiable at c and continuous in $(c-\delta,\ c+\delta)$, where $\delta > 0$;*

(ii) *$f(x,\ y)$ is defined over the set S of points $(x,\ y)$ satisfying $c-\delta < y < c+\delta$ and $a(y) \leqslant x \leqslant b(y)$; and is continuous at $(a(c),\ c)$ and $(b(c), c)$;*

(iii) *for each y in $(c-\delta,\ c+\delta)$, $\int_{a(y)}^{b(y)} f(x,\ y)dx$ exists as an L-integral;*

(iv) *$P(x,h) = (f(x,\ c+h) - f(x,\ c))/h \to \phi(x,c)$ as $h \to 0$, p.p. in $(a(c),\ b(c))$; and*

(v) *a certain summable function $g(x)$ exists in $(a(c),\ b(c))$, satisfying $|P(x,h)| \leqslant g(x)$ for every point $(x, c+h)$ of S satisfying $h \neq 0$, and $a(c) < x < b(c)$.*

Then the function of y, $\int_{a(y)}^{b(y)} f(x,\ y)dx$ has a derivative at c equal to

$$\int_{a(c)}^{b(c)} \phi(x,c)\,dx + b'(c)\,f(b(c),c) - a'(c)\,f(a(c),c).$$

PROOF. Let $F(y)$ denote $\int_{a(y)}^{b(y)} f(x, y)\, dx$.

CASE 1. $a(c) = b(c)$. Then,

$$(F(c+h) - F(c))/h = \frac{1}{h} \int_{a(c+h)}^{b(c+h)} f(x, c+h)\, dx$$

$$= \frac{1}{h}\Big[(b(c+h) - b(c)) - (a(c+h) - a(c))\Big] \times$$

$$\times f(a(c), c) + \alpha(h)$$

where $\alpha(h) \to 0$ as $h \to 0$ on account of (ii). Hence the right side tends to $(b'(c) - a'(c))\, f(a(c), c)$. This completes the proof in this case.

CASE 2. $a(c) < b(c)$. Let

$$M(h) = \max_{|t|<|h|} a(c+t) \text{ and } m(h) = \min_{|t|<|h|} b(c+t),$$

both of which exist on account of (i). We observe that $M(h) \to a(c)$ and $m(h) \to b(c)$ as $h \to 0$. Hence a positive δ exists such that, for $|h| < \delta$, $M(h) < m(h)$. Hence for $0 < |h| < \delta$,

$$(F(c+h) - F(c))/h = \nu_1(h) + \nu_2(h) + \nu_3(h),$$

where

$$\nu_1(h) = \int_{M(h)}^{m(h)} [f(x, c+h) - f(x, c)]h^{-1}\, dx,$$

$$\nu_2(h) = \Big[\int_{m(h)}^{b(c+h)} f(x, c+h)\, dx - \int_{m(h)}^{b(c)} f(x, c)\, dx\Big]h^{-1},$$

and

$$\nu_3(h) = \Big[\int_{a(c+h)}^{M(h)} f(x, c+h)\, dx - \int_{a(c)}^{M(h)} f(x, c)\, dx\Big]h^{-1}.$$

Now as $h \to 0$, $\nu_1(h) \to \int_{a(c)}^{b(c)} \phi(x, c)\, dx$ by Lemma I below, and $\nu_2(h) \to b'(c)\, f(b(c), c)$, $\nu_3(h) \to -a'(c) f(a(c), c)$, both by Lemma II below.

This proves the theorem save for the lemmas.

LEMMA I. $\nu_1(h) \to \int_{a(c)}^{b(c)} \phi(x, c)\, dx$ *as* $h \to 0$.

PROOF. Define $\chi(x, h)$ in $(a(c), b(c))$ as follows:

$$\chi(x, h) = (f(x, c+h) - f(x, c))/h \text{ for } M(h) \leqslant x \leqslant m(h);$$
$$= 0, \text{ otherwise.}$$

Obviously $|\chi(x, h)| \leqslant g(x)$ by (v). We observe that $\chi(x, h) \to \phi(x, c)$ as $h \to 0, p.p.$ in $(a(c), b(c))$ by (iv). Further $\nu_1(h) = \int_{a(c)}^{b(c)} \chi(x, h)\, dx$ tends to $\int_{a(c)}^{b(c)} \phi(x, c)\, dx$ as h tends to zero by an argument similar to that given in [1, pp. 67-68].

LEMMA II. (1) $\nu_2(h) \to b'(c) f(b(c), c)$ *and*

(2) $\nu_3(h) \to -a'(c) f(a(c), c)$ *as* $h \to 0$.

PROOF OF (1). Writing

$$f(x, c+h) = f(b(c), c) + u_1(x, h)$$

in $m(h) \leqslant x \leqslant b(c+h)$, and

$$f(x, c) = f(b(c), c) + u_2(x, h)$$

in $m(h) \leqslant x \leqslant b(c)$, we have

$$\nu_2(h) = f(b(c), c)\,(b(c+h) - b(c))/h + \left[\int_{m(h)}^{b(c+h)} u_1(x, h)\, dx\right]/h - \left[\int_{m(h)}^{b(c)} u_2(x, h)\, dx\right]/h.$$

By continuity of $f(x, y)$ at $(b(c), c)$, provided by hypothesis (ii) of the theorem, $u_1(x, h)$ and $u_2(x, h)$ approach zero, and by Lemma III, below, both $(b(c+h) - m(h))/h$ and $(b(c) - m(h))/h$ are bounded, as $h \to 0$.

PROOF OF (2). Similar to the proof given above.

LEMMA III. *As $h \to 0$, the four functions*

(1) $(M(h) - a(c+h))/h$, (2) $(M(h) - a(c))/h$,

(3) $(b(c) - m(h))/h$, (4) $(b(c) + c - m(h))/h$

are bounded.

PROOF OF (1). We have

$$(M(h) - a(c+h))/h = (M(h) - a(c))/h + (a(c) - a(c+h))/h.$$

The boundedness of the first term on the right side is proved below, that of the second follows from the existence of $a'(c)$.

PROOF OF (2). We have in virtue of the hypothesis (i) of the theorem, $M(h) = a(c + \mu h)$ where $|\mu| \leqslant 1$, and

$$M(h) - a(c) = \mu h\,(a'(c + v(h))$$

where $v(h) \to 0$ as $h \to 0$. Hence $(M(h) - a(c))/h = \mu(a'(c) + v(h))$ which is bounded as $h \to 0$.

PROOF OF (3). Similar to (2).

PROOF OF (4). Similar to (1).

Thus the theorem is completely proved.

My thanks are due to Dr. V. Ramaswami for his valuable criticism.

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MATHEMATICAL NOTES

Moments of a certain distribution

By A. R. KAMAT, *Fergusson College, Poona*

1. In a paper recently prepared for publication (to appear in *Biometrika*) the author has discussed a non-parametric test based on the ranking of observations. This test essentially depends on the probability distribution of the statistic $D_{n,m}$ defined by

$$D_{n,m} = R_n - R_m + m \tag{1}$$

where R_n and R_m are the ranges of the two parts resulting from a random partition of the integers

$$1, 2, 3, \ldots, m+n \tag{2}$$

into two parts containing n and m integers respectively. We are giving below a method of derivation of the moments of $D_{n,m}$ which is interesting in itself as an application of combinatorial methods.

2. The distribution $D_{n,m}$ has been obtained in the paper mentioned above from the joint distribution of R_n and R_m which is also derived therein. Since it is not easy to find the moments of $D_{n,m}$ from its distribution directly we shall use the joint distribution of R_n and R_m for this purpose. Write

$$R_n = n - 1 + i, \quad R_m = m - 1 + j, \quad \text{and} \quad K = \binom{m+n}{n},$$

where $i = 0, 1, 2, \ldots, m$, and $j = 0, 1, 2, \ldots, n$. Then the joint distribution of R_n and R_m has been shown to be as follows:

(i) $P(R_n = n-1,\ R_m = m-1) = 2K^{-1}$;

(ii) $P(R_n = n-1+i,\ R = m-1+n)$

$$= (m+1-i)\binom{n+i-2}{n-2}K^{-1},\ i = 0, 1, 2, \ldots, n-1;$$

(iii) $P(R_n = n-1+m,\ R = m-1+j)$

$$= (n+1+j)\binom{m+j-2}{m-2}K^{-1},\ j = 0, 1, 2, \ldots, m-1;$$

$$\text{(iv)}\quad P(R_n = n-1+i,\ R_m = m-1+j) = 2\binom{i+j-2}{j-1}K^{-1};$$

$$i = 1, 2, 3, \ldots, m-1 \text{ and } j = 1, 2, 3, \ldots, n-1. \tag{3}$$

It can be proved from (3), or independently, that the (marginal) distribution of R_n is

$$P(R_n = n-1+i) = (m+1-i)\binom{n+i-2}{n-2}K^{-1}, \tag{4}$$

where $i = 0, 1, 2, \ldots, m$ (see the author's paper, to appear in *J. American Stat. Association*).

3. The derivation of moments depends on the successive application of the following combinatorial formula of the binomial coefficients :

$$\sum_{k=0}^{b-1}\binom{a+k}{a} = \binom{a+b}{a+1}. \tag{5}$$

Taking the origin at $R_n = n-1$, $R_m = m-1$, the joint factorial moment $\mu_{[r,0]}$ or, what is the same, the factorial moment $\epsilon(i^{(r)})$ of R_n, can be obtained with the help of (5). It is

$$\mu_{[r,0]} = \frac{m^{(r)}n^{(2)}}{(n+r)^{(2)}}, \tag{6}$$

where $m^{(r)}$ denotes as usual $m!/(m-r)!$.

4. The cross factorial moments $\mu_{[r,s]} = \epsilon(i^{(r)}j^{(s)})$ are much more complicated. The application of the formula (5) however, after some algebra, reduces the factorial moment to the form

$$\mu_{[r,s]} = \frac{n^{(s)}\,n^{(2)}\,m^{(r+2)}}{(m+n)^{(2)}\,(n+r)^{(2)}} + \frac{m^{(r)}\,m^{(2)}\,n^{(s+2)}}{(m+n)^{(2)}\,(m+s)^{(2)}} + \frac{2\,L_{r,s}}{\binom{m+n}{n}}, \tag{7}$$

where

$$L_{r,s} = \sum_{i=i}^{m-1}\sum_{j=1}^{n-1} i^{(r)}\ j^{(s)}\binom{i+j-2}{i-1}, \tag{8}$$

and this reduces to

$$\sum_{i=1}^{m-1}\left[ns+(i-1)(n-1)\right].\frac{i^{(r)}(i+s-2)!\,(n+i-2)!}{(i-1)!\,(i+s)!\,(n-s-1)!}. \tag{9}$$

Now for a given $s \geqslant 1$ we may simplify (9) as follows. Write

$$[ns+(i-1)(n-1)]\frac{i^{(r)}(i+s-2)!}{(i-1)!}=[ns+(i-1)(n-1)]i^{(r)}i^{(s-1)}$$

and express it in the form of a series of factorials $(i+s)^{(t)}$, $(t=0, 1, 2, \ldots, r+s)$ as

$$\sum_{t=0}^{r+s} K_t(n,r,s)\ (i+s)^{(t)}$$

where the coefficients K_t will depend on n, r, s and can be evaluated. Then $L_{r,s}$ of (9) becomes

$$\begin{aligned} L_{r,s} &= \sum_{t=0}^{r+s} K_t(n,r,s) \sum_{i=1}^{m-1} \frac{(n+i-2)!}{(n-s-1)!\,(i+s-t)!} \\ &= \sum_{t=0}^{r+s} K_t(n,r,s) \frac{(n+t-s-2)!}{(n-s-1)!} \left\{\binom{n+m-2}{n+t-s-1} - \right. \\ &\qquad\qquad \left. - \binom{n-1}{n+t-s-1}\right\}. \end{aligned} \tag{10}$$

For instance when $r=s=1$, $K_0=n-2$, $K_1=-(n-2)$ and $K_2=n-1$, and therefore

$$\begin{aligned} L_{1,1} &= \sum_{t=0}^{2} K_t \frac{(n+t-3)!}{(n-3)!} \left\{\binom{n+m-2}{n+t-2} - \binom{n-1}{n+t-2}\right\} \\ &= \binom{m+n}{n}\left[\frac{m^{(2)}n^{(2)}}{(m+n)^{(2)}} - \frac{mn}{m+n} + 1\right] - 1 \end{aligned}$$

which gives, after some simplification, the factorial moment

$$\mu_{[1,1]} = 2 - \frac{2}{\binom{m+n}{n}} + \frac{m^{(2)}n^{(2)} - 4\,mn}{(m+1)(n+1)}. \tag{11}$$

5. To find the first four moments of $D_{n,m}$ the following cross factorials are required in addition to (11). They are obtained in a similar manner. Thus we have

$$\mu_{[2,1]} = \frac{4}{\binom{m+n}{n}} - 4 - \frac{m^{(2)} n^{(2)} - 8\,mn}{(m+1)(n+1)} + \frac{m^{(2)} n^{(2)} (mn-2)}{(m+1)(n+2)^{(2)}},$$

$$\mu_{[2,2]} = 16 - \frac{16}{\binom{m+n}{n}} - \frac{32\,mn}{(m+1)(n+1)} + \frac{m^{(2)} n^{(2)} [m^{(2)} n^{(2)} - 8(m-1)(n-1) + 36]}{(m+2)^{(2)} (n+2)^{(2)}},$$

$$\mu_{[3,1]} = 12 - \frac{12}{\binom{m+n}{n}} + \frac{12\,mn\ (m-3)}{(m+1)(n+1)} - \frac{2\,m^{(2)} n^{(2)} (m-1)(n+1)}{(m+1)(n+2)^{(2)}} + \frac{m^{(2)} n^{(2)} (m^{(2)} n^{(2)} - 48) + 2\,m^{(2)} n\,(m^{(2)} n^{(2)} - 72)}{(m+1)(n+3)^{(3)}}. \tag{12}$$

6. From the results (6), (11) and (12) we obtain the following central moments for the joint distribution of R_m and R_n which are necessary in order to evaluate the first four moments of $D_{n,m}$.

$$\mu_{1,0} = \frac{m(n-1)}{n+1}, \quad \mu_{2,0} = \frac{2\,m(n-1)(m+n+1)}{(n+1)(n+2)^{(2)}},$$

$$\mu_{3,0} = -\frac{2\,m(n-1)(n-3)(m+n+1)(2m+n+1)}{(n+1)^{(2)} (n+3)^{(3)}},$$

$$\mu_{4,1} = \frac{2\,m(n-1)(m+n+1)}{(n+1)^{(3)} (n+4)^{(4)}} [12(m+1)(m+n)(n^{(2)} - 2n + 3) + n^{(4)} - 21\,n^{(3)} + 15\,n^{(2)} - 23\,n + 12], \tag{13}$$

and the cross moments

$$\mu_{1,1} = 2 - \frac{2}{\binom{m+n}{n}} - \frac{4\,mn}{(m+1)(n+1)},$$

$$\mu_{2,1} = -2 - \frac{2}{\binom{m+n}{n}} \left[1 + \frac{2\,m(n-1)}{n+1}\right] + \frac{4\,mn}{(m+1)(n+1)} - \frac{8\,m(n-1)(m+n+1)}{(m+1)(n+1)(n+2)^{(2)}},$$

$$\mu_{2,2} = 10 - \frac{10}{\binom{m+n}{n}} -$$

$$- \frac{4[2\, m^{(2)} n^{(2)} + n\,(n+1)\,(m-1) + m(m+1)\,(n-1)]}{(m+1)\,(n+1)\binom{m+n}{n}} +$$

$$+ \frac{4\,A}{(m+1)\,(n+1)\,(m+2)^{(2)}\,(n+2)^{(2)}},$$

where

$$A = m^2 n^2 (m+n+1)\,(m+n-14) + mn\,(m+n)^{(2)}\,(m+n-3) -$$

$$- 2\,(m+n+2)^{(3)}\,(m+n+1) - 2\,m\,n\,(m^2 n^2 + 2\,m\,n - 4).$$

$$\mu_{3,1} = 2 - \frac{2}{\binom{m+n}{n}} - \frac{6\,m(n-1)\,[m(n-1)+n+1]}{(n+1)^{(2)}\binom{m+n}{n}} -$$

$$- \frac{16\,mn}{(m+n)\,(n+1)} + \frac{24\,m^2 n}{(m+1)\,(n+1)^2} - \frac{36\,m^3 n}{(m+1)\,(n+1)^3} +$$

$$+ \frac{24\,m^2\,(n-1)^2\,(m+n+1)}{(m+1)\,(n+1)^2\,(n+2)^2} +$$

$$+ \frac{12 m(n+n+1)}{(m+1)\,(n+1)\,(n+3)^3} \times$$

$$\times\,(n^4 + 9\,n^3 + 14\,n^2 + 3\,n - 3 - 6\,mn^3 - 16\,m\,n^2 -$$

$$- 11\,mn^2 - 3m). \quad (14)$$

7. From these joint moments it is now easy to obtain the first four moments of $D_{n,m}$. They are:

$$\mu_1(D) = n + \frac{m(n-1)}{n+1} + \frac{n(m-1)}{m+1} = m - \frac{2\,m}{1+n} + \frac{2\,n}{m+1},$$

$$\mu_2(D) = \mu_{20} - 2\,\mu_{11} + \mu_{02}$$

$$= \frac{2\,m(n-1)\,(m+n+1)}{(n+1)\,(n+2)^{(2)}} + \frac{2\,n(m-1)\,(m+n+1)}{(m+1)\,(m+2)^{(2)}} +$$

$$+ \frac{8\,mn}{(m+1)\,(n+1)} + \frac{4}{\binom{m+n}{n}} - 4,$$

$$\mu_3(D) = \mu_{30} - 3\,\mu_{21} + 3\,\mu_{12} - \mu_{03}$$

$$= \frac{2\,n(m-1)\,(m-3)\,(m+n+1)\,(m+2\,n+1)}{(m+1)^2\,(m+3)^{(3)}} -$$

$$- \frac{2\,m(n-1)\,(n-3)\,(m+n+1)\,(2\,m+n+1)}{(n+1)^2\,(n+3)^{(3)}} +$$

$$+ \frac{24\,(m-n)\,(m-1)\,(n-1)\,[(m+n+2)^{(2)} - mn]}{(m+1)\,(n+1)\,(m+2)^{(2)}\,(n+2)^{(2)}} +$$

$$+ \frac{12\,(m-n)}{(m+1)\,(n+1)} \left[\frac{4\,mn}{(m+1)\,(n+1)} - 2 - \frac{(m-1)\,(n-1) - 2}{\binom{m+n}{n}}\right], \tag{15}$$

and

$$\mu_4(D) = \mu_{40} - 4\,\mu_{31} + 6\,\mu_{22} - 4\,\mu_{13} + \mu_{04}, \tag{16}$$

where the values of the various joint moments are to be substituted. We do not reproduce the final expression for $\mu_4(D)$ because of its length since it could not be simplified very much.

8. For the symmetrical case $m = n$ these expressions simplify and become

$$\mu_1(D) = n, \ \mu_2(D) = \frac{4\,n(4\,n-1)}{(n+1)\,(n+2)} - 4 - \frac{4}{\binom{2n}{n}}, \quad \mu_3(D) = 0,$$

and

$$\mu_4(D) = 2\,\mu_{40} - 8\,\mu_{31} + 6\,\mu_{22}, \tag{17}$$

where $m = n$ is to be substituted in the moments given above in (13) and (14). The simplified expressions not reproduced here on account of its length.

9. For the limiting and other properties of R_m and $D_{n,m}$ distributions as well as their statistical significance and applications the reader may refer to the papers mentioned.

On generalized Taylor expansions

By K. S. PADMANABHAN, *Annamalai University, Annamalainagar.*

1. Introduction. The aim of this paper is to expand any function $f(x)$ analytic at zero in the form $\sum_{n=0}^{\infty} t_n F_n(x)$ where the functions $F_n(x)$ are given by

$$F_n(x) = x^n + \sum_{p=1}^{\infty} a_{n,p}\, x^{n+p}. \tag{1}$$

The problem has been discussed when $f(x)$ and $F_n(x)$ are all *entire* functions, by V. Ganapathi Iyer [On the space of integral functions II, *Quart. J. Math.* (Oxford) (1950), 93].

2. I prove the following result.

THEOREM. *Let* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *and* $|a_n| \leqslant K^n$ *for* $n = 1, 2, \ldots,$ *with* $|a_0| \leqslant K$. *Besides let* $|a_{n,p}| \leqslant K^p$, $n = 0, 1, 2, \ldots; p = 1, 2, \ldots$. *Then the function* $f(x)$ *has an absolutely and uniformly convergent expansion of the form* $\sum_{n=0}^{\infty} t_n F_n(x)$ *valid in* $|x| < 1/2K$ *and the expansion is unique.*

PROOF. Let us define (t_n), $n \geqslant 0$ by the equations

$$t_0 = a_0,\ t_n + t_{n-1}\, a_{n-1,1} + t_{n-2}\, a_{n-2,2} + \ldots + t_0 a_{0,n} = a_n \text{ for } n \geqslant 1. \tag{2}$$

The equations (2) determine t_n successively and uniquely. Now,

$$|t_0| = |a_0| \leqslant K,\ |t_1| \leqslant |t_0|\,|a_{0,1}| + |a_1| \leqslant K^2 + K = K(1+K).$$

I shall prove by induction that for $n \geqslant 1$

$$|t_n| \leqslant 2^{n-1} K^n (1+K). \tag{3}$$

Now (3) is found to be true for $n = 1$. Assume it has been proved for $n = 1, 2, \ldots, (m-1)$. Then from (2) we get

$$|t_m| \leqslant |t_{m-1}|\,|a_{m-1,1}| + |t_{m-2}|\,|a_{m-2,2}| + \ldots + |t_0|\,|a_0 m| + |a_m|.$$

Therefore

$$|t_m| \leqslant 2^{m-2} K^{m-1} (1+K) K + 2^{m-3} K^{m-2} (1+K) K^2 + \ldots +$$
$$+ \ldots + K(1+K) K^{m-1} + K^{m+1} + K^m$$
$$= K^m (1+K)\, 2^{m-1}.$$

Hence (3) is true. And

$$F_n(x) = \sum_{p=1}^{\infty} a_{n,p}\, x^{n+p}$$

$$|F_n(x)| = |x|^n \left\{ 1 + \sum_{p=1}^{\infty} |a_{n,p}|\; |x|^p \right\}$$
$$\leqslant |x|^n \left\{ 1 + \sum_{p=1}^{\infty} K^p\, |x|^p \right\}$$
$$= |x|^n \left\{ 1 + \frac{K|x|}{1 - K|x|} \right\},$$

provided that $|x| < 1/K$. Again, using (3), we get

$$|t_n F_n(x)| \leqslant 2^{n-1} K^n (1+K) \left\{ 1 + \frac{K|x|}{1-K|x|} \right\} |x|^n$$
$$= \tfrac{1}{2}(1+K) \left(1 + \frac{K|x|}{1-K|x|} \right) (2K|x|)^n.$$

Hence $\sum_{n=0}^{\infty} t_n F_n(x)$ is dominated for $|x| < 1/K$ by the series

$$\sum_{n=0}^{\infty} \tfrac{1}{2}(1+K) \left(1 + \frac{K|x|}{1-K|x|} \right) 2^n K^n |x|^n$$

which, in turn, converges provided that $2K|x| < 1$. Hence the series $\sum_{n=0}^{\infty} t_n F_n(x)$ converges absolutely and uniformly for $|x| < 1/2K$ to an analytic function $F(x)$, say. Now, by Weirstrass' theorem, the coefficients of different powers of x in $F(x)$ can be got by expanding $F_n(x)$ and collecting like powers of x. But in virtue of the relations (2), it is easily seen that the coefficient of x^n in $F(x)$ is a_n. Hence $F(x) = f(x)$, and the theorem is established.

3. Remark. The hypotheses $|a_0| \leqslant K$ and $|a_n| \leqslant K_n$ for $n = 1$, $2, \ldots$ can be replaced by the more general hypothesis that the radius of convergence ρ of $f(x) \Sigma a_n x^n$ is not less than $1/K$ and the same conclusion holds.

For, then, $\lim\sup |a_n|^{1/n} = 1/\rho \leqslant K$. Therefore $|a_n| \leqslant (K + \epsilon)_n$ for $n \geqslant n_0$ (say), where ϵ can be made as small as we please. Define $\phi(x) = f(x) - \sum_{r=0}^{n_0-1} a_r x^r$; we see that the above working holds good for $\phi(x)$ instead of $f(x)$ with K replaced by $K + \epsilon$. Hence $\phi(x)$ can be expanded in the required form in $|x| < 1/2(K + \epsilon)$ where ϵ can be made small to our liking. The polynomial $\sum_{r=0}^{n_0=1} a_r x^r$ can also be expressed in the required form, since all its coefficients can be divided by a suitable constant M, so that the new coefficients A_0, $A_1 \ldots A_{n_0-1}$ satisfy the condition, viz. $|A_{n_0-1}| \leqslant K^{n_0-1}$; $|A_{n_0-2}| \leqslant K^{n_0-2}$ and so on and $|A_0| \leqslant K$. Hence $\sum_{r=0}^{n_0-1} A_r x^r$ can be expanded in the desired form and consequently the polynomial $\sum_{r=0}^{n_0-1} a_r x^r$. Hence, the expansion of

$$f(x) = \phi(x) + \sum_{r=0}^{n_0-1} a_r x^r$$

in the form $\sum_{n=0}^{\infty} t_n F_n(x)$ is valid in $|x| < 1/2K$, since ϵ can be chosen to be small.

Allied but more complicated theorems have been discussed by G. S. Ketchum [On certain generalizations of the Cauchy-Taylor expansion theory, *Trans. American Math. Soc.* 40 (1936), p. 208].

Finally, I should thank Dr. V. Ganapathi Iyer for his helpful guidance in preparing this paper.

CLASSROOM NOTES

Applications of Fermat's theorem

By R. Ramachandra

It is a well-known unproved conjecture due to Euler that the diophantine equation $x^4 + y^4 + z^4 = u^4$ has no rational solutions apart from the trivial solution $x = y = z = u = 0$. The values of the prime p so that the congruence $x^n + y^n + z^n \equiv 0(p)$ shall have solutions when $n = 4$, 6 and 8, have been discussed by G. Carnacchia. The following theorem, though a very easy consequence of Fermat's theorem, gives, as immediate corollaries, many interesting theorems like the impossibility of rational solutions of the equation

$$ax^{2m} + by^{2m} + cz^{2m} = Au^g, \ (2m, g) > 1,$$

in certain cases.

Theorem. *If $k \geqslant 1$ be any divisor of m, and p a prime number of the form $2k(2n + 1) + 1$, and none of the integers a, b, c, $b^{2n+1} + c^{2n+1}$, $c^{2n+1} + a^{2n+1}$, $a^{2n+1} + b^{2n+1}$, and the circulant*

$$C = \left[a^{2n+1} + b^{2n+1} + c^{2n+1}, \binom{2n+1}{1} a^{2n}b, \binom{2n+1}{2} a^{2n-1}b^2, \ldots, \binom{2n+1}{2n} ab^{2n} \right]$$

are divisible by p, then the relation

$$ax^{2m} + by^{2m} + cz^{2m} = p^\alpha Q \quad (\alpha \geqslant 1, Q, x, y, z \text{ integers}) \qquad (1)$$

implies that x, y, z are all divisible by p, and if $Q \not\equiv 0(p)$ then $\alpha \equiv 0(2m)$.

Proof. If two of the integers x, y, z are divisible by p, then the third is also divisible by p. Hence the integers x, y, z are all divisible by p or at least two of them are not divisible by p. We shall show that the latter case is impossible. To this end we suppose that

(i) $p \mid x; p \nmid yz$. We then have $by^{2m} + cz^{2m} \equiv 0(p)$, i.e. $by^{2m} \equiv -cz^{2m}(p)$. Here taking the $(2n+1)$th powers on both the sides, and observing that

$$y^{2m(2n+1)} \equiv \{y^{m/k}\}^{2k(2n+1)} \equiv 1(p), \text{ and } z^{2m(2n+1)} \equiv 1(p),$$

we are led to $b^{2n+1} \equiv -c^{2n+1}(p)$, a contradiction to the assumption that $b^{2n+1} + c^{2n+1} \not\equiv 0(p)$.

Again, we suppose that

(ii) $p \nmid xyz$. Let y_1 be determined by the congruence $yy_1 \equiv 1(p)$. Multiplying (1) by y_1^{2m}, and writing $\sigma = (xy_1)^{2m}$ and $t = (zy_1)^{2m}$, we obtain on transposing, the relation

$$a\sigma + b \equiv -ct(p), \tag{2}$$

where, obviously, σ^{k}, t^{k} are integers prime to p, and hence

$$\sigma^{2n+1} \equiv t^{2n+1} \equiv 1(p) \tag{3}$$

by Fermat's theorem. Raising both sides of the congruence (2) to the $(2n+1)$th power and transposing,

$$(a^{2n+1} + b^{2n+1} + c^{2n+1}) + \binom{2n+1}{1} a^{2n}\, b\sigma^{2n} +$$
$$+ \binom{2n+1}{2} a^{2n-1} b^2\, \sigma^{2n-1} + \dots + \binom{2n+1}{2n} ab^{2n}\, \sigma \equiv 0(p). \tag{4a}$$

Multiplying this in succession by $\sigma, \sigma^2, \dots, \sigma^{2n}$, we obtain, on making use of (3), the congruence system

$$\left.\begin{aligned}
&\binom{2n+1}{1} a^{2n} b + \binom{2n+1}{2} a^{2n-1} b^2\, \sigma^{2n} + \dots + \\
&\qquad\qquad + \sigma(a^{2n+1} + b^{2n+1} + c^{2n+1}) \equiv 0, \\
&\binom{2n+1}{2} a^{2n-1} b^2 + \binom{2n+1}{3} a^{2n-2} b^3\, \sigma^{2n} + \dots + \\
&\qquad\qquad + \sigma \binom{2n+1}{1} a^{2n} b \equiv 0, \\
&\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\
&\binom{2n+1}{2n} ab^{2n} + (a^{2n+1} + b^{2n+1} + c^{2n+1})\, \sigma^{2n} + \dots + \\
&\qquad\qquad + \binom{2n+1}{2n-1} a^2 b^{2n-1}\, \sigma \equiv 0,
\end{aligned}\right\} \tag{4b}$$

(4a) and (4b), on multiplication by suitable co-factors of the circulant C, and addition, yield the congruence $C \equiv 0(p)$ in contradiction to the assumptions made. Hence it follows that x, y, z are all divisible by p.

If $Q \not\equiv 0(p)$ then at least one of the integers x, y, z is different from zero. In this case, we see, on removing the H.C.F. of x, y, z from both sides of the relation (1), that $\alpha \equiv 0(2m)$. This completes the proof of the theorem.

COROLLARY 1. *If a, b, c and p satisfy the conditions of the theorem, then $ax^{2m} + by^{2m} + cz^{2m}$ is different from zero, and the highest power of the prime p which divides this expression is $p^{2m\beta}$, where β is a non-negative integer.*

COROLLARY 2. *If a, b, c and p satisfy the conditions of the theorem, then the only rational solution of either of the diophantine equations*

$$\left.\begin{array}{l} ax^{2m} + by^{2m} + cz^{2m} = p^{\alpha}Au^{g}; \quad (2m, g) > 1, \quad (2m, g) \nmid \alpha, \\ ax^{2m} + by^{2m} + cz^{2m} = p^{\alpha}A(au^{2m} + bv^{2m} + cw^{2m}); \; 2m \nmid \alpha, \end{array}\right\} \qquad (5)$$

taken severally, is the trivial solution in which the unknowns involved are all equal to zero provided that $p \nmid A$.

The proof of Corollary 2 follows by substituting rational values, assuming them to exist, of the unknowns involved in the equations, clearing the fractions, and finally equating the highest powers of the prime p on both sides. This easily leads to a contradiction proving the corollary.

A number of special cases follows from Corrollary 2, e.g.

$$a = b = c = \alpha = A = 1, \; k = 2, \; n = 3$$

gives the impossibility of rational solutions of

$$\left.\begin{array}{l} x^4 + y^4 + z^4 = 29u^2 \\ \text{and} \\ x^4 + y^4 + z^4 = 29(u^4 + v^4 + w^4), \end{array}\right\} \qquad (6)$$

except for the trivial solution in which the unknowns involved are all equal to zero.

In conclusion, I wish to thank Professor B. S. Madhava Rao, for his encouragement and interest in this work.

Proof of a theorem on the product of matrices

By PAVMAN MURTHY, *Osmania University*

THE following is a theorem on the rank of the product of two matrices which is quite well known. The proof of the theorem given here is based on the conception of linear dependence and independence of vectors and appears to be new.

It is proved in different textbooks on matrices that the rank of a matrix is equal to the maximal number of linearly independent row vectors or column vectors. The following proof is based on this fact.

THEOREM. *The rank of the product of two matrices cannot exceed the rank of either of its factors.*

PROOF. Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdot & \cdot & a_{mn} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & \cdot & \cdot & b_{1k} \\ b_{21} & b_{22} & \cdot & \cdot & b_{2k} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{n1} & b_{n2} & \cdot & \cdot & b_{nk} \end{bmatrix}.$$

be two matrices so that the product AB exists. Let the rank of AB be R and those of A and B be r_1 and r_2 respectively. Let the row vectors of A be denoted by $X_i (i = 1, 2, \ldots m)$ where $X_i = (a_{i1}, a_{i2}, \ldots, a_{in})$ and the column vectors of B be $Y_i \, (i = 1, 2, \ldots, k)$ where

$$Y_i = \begin{bmatrix} b_{1i} \\ b_{2i} \\ \cdot \\ \cdot \\ \cdot \\ b_{ni} \end{bmatrix},$$

so that

$$AB = \begin{bmatrix} X_1Y_1 & X_1Y_2 & X_1Y_3 & \cdot & X_1Y_k \\ X_2Y_1 & X_2Y_2 & X_2Y_3 & \cdot & X_2Y_k \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ X_mY_1 & X_mY_2 & X_mY_3 & \cdot & X_mY_k \end{bmatrix}$$

where $X_i \, Y_j$ denotes the scalar product of vectors X_i and Y_j. If possible suppose that $R > r_1$. Then any R row vectors of A are linearly dependent. Therefore we can choose numbers α_1, α_2, $\alpha_3, \ldots, \alpha_R$, not all the α_s zero, such that $\sum_{i=1}^{R} \alpha_i X_i = 0$ for any R row vectors X_i of A. Then the vector

$$(Y_1 \sum_{i=1}^{R} \alpha_i X_i, \; Y_2 \sum_{i=1}^{R} \alpha_i X_i, \ldots \; Y_k \sum_{i=1}^{R} \alpha_i X_i) = (0),$$

i.e. $\sum_{i=1}^{R} \alpha_i (X_i Y_1, \; X_i Y_2, \ldots, \; X_i Y_k) = (0)$, for any R row vectors of AB, i.e. any R row vectors of AB are linearly dependent. This contradicts the hypothesis that R is the rank of AB. Therefore $R \leqslant r_1$.

We can repeat the same argument taking the column vectors of B and AB to get $R \leqslant r_2$.

It may be noted that the same method is used in the treatment of the rank of a module (cf. B. L. Van der Waerden, *Modern Algebra*, Vol. 2).

On the sequence $\{V_n\}$, $V_n = \alpha V_{n-1} + \beta V_{n-2}$

By M. PERISASTRI, *M. R. College, Vizianagaram*

It was proved by K. Subbarao (*American Math. Monthly*, 50, 1953 p. 682) that if $\{U_n\}$ be a sequence of positive integers defined by $U_n = U_{n-1} + U_{n-2}$, where $U_1 = U_2 = 1$ and k an integer > 1, then an integer l, $0 < l < k$, exists such that

(i) $$U_{\sum_{i=1}^{k} n_i - l + 1} < \prod_{i=1}^{k} U_{n_i} < U_{\sum_{i=1}^{k} n_i - l + 1}$$

for $n_i > n_0(k)$, $i = 1, 2, \ldots k$; and

(ii) there are exactly mk members of the sequence between U_n^k and U_{n+m}^k.

In fact results like (i) and (ii) stated above are true for any sequence $\{V_n\}$ of positive real numbers where $V_n = \alpha V_{n-1} + \beta V_{n-2}$. Now I prove the following

THEOREM. *If $\{V_n\}$ be a sequence of positive real numbers defined by $V_n = \alpha V_{n-1} + \beta V_{n-2}$ where V_1, V_2 are defined, α, $\beta > 0$ and independent of n, $0 < r = (\alpha + \sqrt{(\alpha^2 + 4\beta)})/2 \neq 1$ and k an integer > 1, then an integer l exists such that*

(1) $$\prod_{i=1}^{k} V_{n_i} \textit{ lies between } V_{\sum_{i=1}^{k} n_i - l} \textit{ and } V_{\sum_{i=1}^{k} n_i - l + 1}$$

for $n_i > n_0(k)$, $i = 1, 2, \ldots, k$; *and*

(2) *there are exactly mk members of the sequence* $\{V_k\}$ *lying between* V_n^k *and* V_{n+m}^k *for* $n > n_0(k)$.

Proof of (1). It is well known from difference equations that if $V_n = \alpha V_{n-1} + \beta V_{n-2}$, then $V_n = A r^n + B s^n$ where r and s are the roots of $x^2 - \alpha x - \beta = 0$, so that $r = (\alpha + \sqrt{(\alpha^2 + 4\beta)})/2$ and $S = (\alpha - \sqrt{(\alpha^2 + 4\beta)})/2$. But $V_1 = A r + Bs$ and $V_2 = A r^2 + B s^2$; we have therefore $A = (V_2 - V_1 s)/r(r-s)$ and $B = (V_2 - V_1 r)/s(s-r)$. Clearly $A > 0$, as $s < 0$.

Now let l be an integer defined by

$$\frac{(1-k)\log A}{\log r} \leqslant l < \frac{(1-k)\log A}{\log r} + 1. \qquad (*)$$

Then for $i = 1, 2, \ldots, k$,

$$\lim_{n_i \to \infty} \frac{\prod_{i=1}^{k} V_{n_i}}{V_{\sum_{i=1}^{k} n_i - l + 1}} = \lim_{n_i \to \infty} \frac{\prod_{i=1}^{k} (A r^{n_i} + B s^{n_i})}{\left(A r^{\sum_{i=1}^{k} n_i - l + 1} + B s^{\sum_{i=1}^{k} n_i - l + 1}\right)}$$

$$= \lim_{n_i \to \infty} \frac{\prod_{i=1}^{k} r^{n_i}[A + B(s/r)^{n_i}]}{r^{\sum_{i=1}^{k} n_i - l + 1}\left[A + B(s/r)^{\sum_{i=1}^{k} n_i - l + 1}\right]}$$

$$= A^{k-1} r^{l-1},$$

since $|s/r| < 1$. But $A^{k-1} r^{l-1} \lesseqgtr 1$ according as $r \gtreqless 1$ by virtue of (*).

Hence $\prod_{i=1}^{k} V_{n_i} \lessgtr V_{\sum_{i=1}^{k} n_i - l + 1}$ according as $r \gtrless 1$ for $n_i > f(k)$, $i = 1, 2, \ldots, k$. Similarly we can show that $\prod_{i=1}^{k} V_{n_i} \gtrless V_{\sum_{i=1}^{k} n_i - l}$, for $n_i > g(k)$, $i = 1, 2, \ldots, k$. Therefore we have

$$V_{\sum_{i=1}^{k} n_i - l} \lessgtr \prod_{i=1}^{k} V_{n_i} \lessgtr V_{\sum_{i=1}^{k} n_i - l + 1} \text{ for } n_i > n_0(k),\ i = 1, 2, \ldots, k \qquad (**)$$

according as $r \lessgtr 1$, which proves (1).

PROOF OF (2). If in (**) we put $n_1 = n_2 = \dots = n_k = n$ it follows that there are exactly k members of the sequence $\{V_n\}$ between V_k^n and V_{n+1}^k for $n > n_0\ (k)$. By extending this argument we can deduce (2).

On orthogonal projection

By U. V. SATYANARAYANA, *Andhra University, Waltair*

IT is well known that the sum of the squares of the orthogonal projections of the sides of equilateral triangles of given size in a given plane Σ on to another given plane σ is constant. The object of this paper is to prove the converse and to consider some connected results.

NOTATION. Let P be a configuration consisting of a finite or an enumerably infinite number of straight line segments A_mB_m of lengths r_m, $m = 1, 2, \dots$, in a plane Σ_0. Let α_m be the angle between the half-ray A_mB_m and a fixed half-ray in Σ_0, and let Σ_δ denote a plane through L inclined at an angle δ to Σ_0. Let $S(\theta, \delta)$ denote the sum (finite or infinite) of the squares of the orthogonal projections of A_mB_m, $m = 1, 2, \dots$, upon Σ_δ after P has been rotated through an angle θ in Σ_0. We supose that $0 \leqslant \theta < \pi$ since $S(\theta, \delta) = S(\theta + \pi, \delta)$.

In what follows, we shall say that P possesses the property C with respect to a plane Σ_δ if $S(\theta, \delta)$ be finite and independent of θ.

Excluding the trivial case $\delta = 0$, the results proved here are :

THEOREM 1(*a*). *If* $\sum_{m=1}^{\infty} r_m^2 \doteq \infty$, *then* $S(\theta, \delta) = \infty$ *for* $0 < \delta < \pi/2$, *and also* $S(\theta, \pi/2) = \infty$, *except possibly for just one* θ *in the range* $0 \leqslant \theta < \pi$ (*e.g. when each* $\alpha_m m = 0$).

THEOREM 1(b). *A necessary and sufficient condition that the configuration P may have the property C with respect to a plane* Σ_δ *for which* $0 < \delta \leqslant \pi/2$, *is that* $S(\theta, \delta)$ *has the same finite value for three distinct values of* θ *in* $0 \leqslant \theta < \pi$.

THEOREM 2. *The possession of the property C by the configuration P in* Σ_0 *with respect to one plane* Σ_δ *for which* $0 < \delta \leqslant \pi/2$ *implies that P has the property C with respect to every plane.*

THEOREM 3. *Given any* $n-1$ *points* $A_1, A_2, \ldots, A_{n-1}$ *in* Σ_0 *such that* A_1 *is distinct from* A_{n-1}, *then either* (i) *the polygon* $A_1A_2\ldots A_{n-1}A_1$ *itself has the property C with respect to a plane* Σ_δ *or,* (ii) *there exists at least one point* A_n*—and in general there are two points—such that the polygon* $A_1A_2\ldots A_nA_1$ *has the property C with respect to* Σ_δ.

COROLLARY. *The possession of the property C by a triangle in* Σ_0 *with respect to a plane* Σ_δ, *for which* $0 < \delta \leqslant \pi/2$, *implies that the triangle is equilateral.*

We shall first prove two lemmas.

LEMMA 1. *Let* $\{r_m\}$ *be a sequence of positive numbers and* $\{\alpha_m\}$ *a sequence of non-negative numbers such that*

$$f(\theta) = \sum_{m=1}^{\infty} r_m^2 \cos^2(\alpha_m + \theta) = \infty$$

for one value of θ *and is finite for* $\theta = \theta_1$ *and* $\theta = \theta_2$. *Then* $\theta_2 - \theta_1$ *is a multiple of* π.

PROOF. Let $\theta_2 = \theta_1 + \phi$ and β_m denote $\alpha_m + \theta_1$. Here

$$\sum_{m=1}^{\infty} r_m^2 \sin^2 \beta_m = +\infty, \tag{1}$$

for otherwise, since $f(\theta_1)$ is finite it would follow that $\sum_{m=1}^{\infty} r_m^2$ itself is finite, and hence that $f(\theta)$ is finite for every θ, contrary to the hypothesis. Also by Hölder's inequality,

$$\sum_{m=1}^{n} | r_m^2 \sin\beta_m \cos\beta_m | \leqslant \left(\sum_{m=1}^{n} r_m^2 \sin^2\beta_m\right)^{1/2} \left(\sum_{m=1}^{n} r_m \cos^2\beta_m\right)^{1/2}$$

$$\leqslant \sqrt{f(\theta_1)} \left(\sum_{m=1}^{n} r_m^2 \sin^2\beta_m\right)^{1/2}.$$

Hence it follows that

$$\sum_{m=1}^{n} r_m^2 \cos^2(\beta_m + \phi)$$

$$= \sum_{m=1}^{n} r_m^2 (\cos^2\beta_m \cos^2\phi - \sin 2\phi \sin\beta_m \cos\beta_m + \sin^2\beta_m \sin^2\phi)$$

$$\geqslant \sin^2\phi \sum_{m=1}^{n} r_m^2 \sin^2\beta_m - | \sin 2\phi | \sqrt{f(\theta_1)} \times \sum_{m=1}^{n} \left(r_m^2 \sin^2\beta_m^2\right)^{1/2} \quad (2)$$

Now let, if possible, $\sin\phi \neq 0$. Then it would follow that

$$f(\theta_2) = \lim_{n\to\infty} \sum_{m=1}^{n} r_m^2 \cos^2(\beta_m + \phi) \geqslant$$

$$\liminf_{n\to\infty} \left(\sum_{m=1}^{n} r_m^2 \sin^2\beta_m\right) \times \left\{\sin^2\phi - \frac{\sqrt{f(\theta_1)}\, | \sin^2\phi |}{\left(\sum_{m=1}^{n} r_m^2 \sin^2\beta_m^2\right)^{\frac{1}{2}}}\right\}, \text{ by (2)},$$

$$= +\infty, \text{by (1)},$$

contrary to the hypothesis that $f(\theta_2)$ is finite. This proves the lemma.

Lemma 2. *If* $\sum_{m=1}^{\infty} r_m^2$ *is finite, a necessary and sufficient condition that the configuration P may possess the property C with respect to a plane* Σ_δ, *where* $0 < \delta \leqslant \pi/2$, *is that* $\sum_{m=1}^{\infty} r_m^2 e^{2i\alpha_m} = 0$.

Proof. It is easily verified that

$$S(\theta, \delta) = \sum_{m=1}^{\infty} \left\{ r_m^2 \cos^2\delta + r_m^2 \sin^2\delta \cos^2(\alpha_m + \theta) \right\}.$$

Now, suppose that P possesses the property C with respect to a plane Σ_δ. Then it follows that $S(\theta, \delta)$ is independent of θ. Hence

$\frac{\partial S}{\partial \theta} = \frac{\partial^2 S}{\partial \theta^2} = 0$ for every θ. Since $\sin \delta \neq 0$ and the series involved are uniformly convergent, we have therefore

$$\sum_{m=1}^{\infty} r_m^2 \sin 2(\alpha_m + \theta) = 0, \quad \sum_{m=1}^{\infty} r_m^2 \cos 2(\alpha_m + \theta) = 0,$$

for every θ. Thus $e^{2i\theta} \sum_{m=1}^{\infty} r_m^2 e^{2i\alpha_m} = 0$ for every θ, and $e^{2i\theta} \neq 0$. The sufficiency follows by retracing the steps.

PROOF OF THEOREM 1(*a*). Here

$$S(\theta, \delta) = \lim_{n \to \infty} \left\{ \cos^2 \delta \sum_{m=1}^{n} r_m^2 + \sin^2 \delta \sum_{m=1}^{n} r_m^2 \cos^2 (\alpha_m + \theta) \right\}$$

Since $\cos \delta \neq 0$ in $0 < \delta < \frac{\pi}{2}$ and $\sum_{m=1}^{\infty} r_m^2$ is $+\infty$, it follows at once that $S(\theta, \delta) = +\infty$ in $0 < \delta < \frac{\pi}{2}$. Also for any θ, one of the numbers $S(\theta, \pi/2)$ and $S(\theta + \pi/2, \pi/2)$ is infinite; for otherwise, it would follow that $\sum_{m=1}^{\infty} r_m^2$ is finite, contrary to the hypothesis. Hence by Lemma 1, there is at most one value of θ in $0 \leqslant \theta < \pi$ for which $S(\theta, \pi/2)$ is finite.

PROOF OF THEOREM 1(*b*). Let $\epsilon_1, \epsilon_2, \epsilon_3$ be the three values of θ for which $S(\epsilon_1, \delta) = S(\epsilon_2, \delta) = S(\epsilon_3, \delta) = \lambda$, say. Here λ is finite by hypothesis and $\sum_{m=1}^{\infty} r_m^2$ is finite by Theorem 1(*a*). Hence on account of the absolute convergence of the series involved we can write $S(\theta, \delta)$ in the form $A \cos^2 \theta - B \sin \theta \cos \theta + C \sin^2 \theta + D$ where

$$A = \sin^2 \delta \sum_{1}^{\infty} r_m^2 \cos^2 \alpha_m, \quad B = \sin^2 \delta \sum_{1}^{\infty} r_m^2 \sin^2 \alpha_m,$$

$$C = \sin^2 \delta \sum_{1}^{\infty} r_m^2 \sin^2 \alpha_m \text{ and } D = \cos^2 \delta \sum_{1}^{\infty} r_m^2.$$

Thus we have

$$(A + D - \lambda) \cos^2 \theta - B \sin \theta \cos \theta + (C + D - \lambda) \sin^2 \theta = 0$$

for three distinct values of $\tan\theta$. This shows that $A+D-\lambda=0$, $B=0$, $C+D-\lambda=0$; and hence $S(\theta, \delta)=\lambda$ for every θ. This proves the sufficiency of the condition. The necessity is obvious.

PROOF OF THEOREM 2. By the hypothesis and Theorem 1 (a) it follows that $\sum_{m=1}^{\infty} r_m^2$ is finite. Now apply Lemma 2.

PROOF OF THEOREM 3. We take $A_m A_{m+1}$ for $A_m B_m$, writing A_{n+1} for A_1. Then the problem is equivalent to settling the existence of two complex numbers $z_1 = r_{n-1}\, e^{i\alpha_{n-1}}$ and $z_2 = r_n e^{i\alpha_n}$ such that

$$r_1 e^{i\alpha_1} + \dots + r_{n-2}\, e^{i\alpha_{n-2}} + z_1 + z_2 = 0$$

and

$$r_1^2\, e^{2i\alpha_1} + \dots + r_{n-2}\, e^{2i\alpha_{n-2}} + z_1^2 + z_2^2 = 0.$$

Thus z_1 and z_2 are to be the roots of the quadratic equation $2z^2 + 2\lambda z + (\lambda^2 + \mu) = 0$, where

$$\lambda = \sum_{m=1}^{n-2} r_m e^{i\alpha_m} \text{ and } \mu = \sum_{m=1}^{n-2} r_m^2\, e^{i\alpha_m}.$$

Here, since $\lambda \neq 0$ (A_1 and A_{n-1} being distinct), $z=0$ cannot be a double root. Denoting the discriminant of the quadratic by Δ, we have the following cases.

CASE (i). $\lambda^2 + \mu = 0$. In this case, one of z_1, z_2 is zero, showing that the polygon $A_1 A_2 \dots A_{n-1} A_1$ itself has the property C with respect to Σ_δ.

CASE (ii). $\Delta = 0$. In this case $z_1 = z_2 \neq 0$, showing that A_n is the mid-point of the segment $A_1 A_{n-1}$. In this case we get just one polygon with the property C with respect to Σ_δ.

CASE (iii). $\lambda^2 + \mu \neq 0$, $\Delta \neq 0$. In this case, z_1 and z_2 are distinct and non-zero and correspondingly we get two distinct points A_n giving rise to two polygons possessing the property C with respect to Σ_δ.

This completes the proof.

For the corollary we need only take $n = 3$ in the above.

Expression for the individual elements of the general conic

By K. SATYANARAYANA

WE employ the usual notation $S \equiv ax^2 + \dots + c = 0$ for the conic; and $A, B, \dots$ for the co-factors of $a, b, \dots$ in

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

In the case of the parabola, i.e. when $C = 0$, setting

$$\rho(X - m) = Gx + Fy;\ \rho(Y - n) = Fx - Gy$$

where

$$\rho > 0,\ \rho^2 = F^2 + G^2 = -\Delta(a + b);$$

$$-\frac{\Delta}{2a} = \frac{\Delta - (a + b)(A + B)}{2m} = \frac{gF - fG}{n} = \rho(a + b),$$

S reduces to

$$(a + b)(Y^2 - 4aX) \tag{1}$$

In the case of the central conic, setting

$$X = \left(x - \frac{G}{C}\right)\cos\theta + \left(y - \frac{F}{C}\right)\sin\theta;$$

$$y = \left(x - \frac{G}{C}\right)\sin\theta - \left(y - \frac{F}{C}\right)\cos\theta,$$

where

$$\sigma > 0,\ \sigma\cos 2\theta = a - b,\ \sigma\sin 2\theta = 2h;$$

$$\frac{\alpha^2}{\sigma - a - b} = -\frac{\beta^2}{\sigma + a + b} = \frac{\Delta}{2c^2};$$

or, what is the same thing,

$$-\alpha^2(\sigma + a + b) = \beta^2(\sigma - a - b) = \frac{2\Delta}{C},$$

S reduces to

$$-\frac{\Delta}{c}\left(\frac{X^2}{\alpha^2}+\frac{Y^2}{\beta^2}-1\right). \tag{2}$$

These transformations of co-ordinates, which reduce $S = 0$ to the standard forms, enable the determination of the expression for any desired element of $S = 0$ from the corresponding expression in the case of the standard form. The following are some of the results which may be obtained thus. In deriving these we make use of the relations

$$\Delta = aA + hH + gG,\ 0 = aG + hF + gC, \text{ and } Bc - F^2 = a\Delta, \text{ etc.}$$

THE PARABOLA

The square of the semi-latus rectum $= -\Delta/(a+b)^3$. (3)

The tangent at the vertex is

$$(a+b)\,[2(Gx+Fy)-(A+B)]+\Delta = 0. \tag{4}$$

The focus is

$$x = -\frac{1}{a+b}\left[g+\frac{A+B}{2\Delta}G\right];\ y = -\frac{1}{a+b}\left[f+\frac{A+B}{2\Delta}F\right], \tag{5.1}$$

or

$$x = \frac{1}{2G}\left[A-\frac{\Delta}{a+b}\right],\ y = \frac{1}{2F}\left[B-\frac{\Delta}{a+b}\right], \tag{5.2}$$

THE CENTRAL CONIC

Assume $\Delta > 0$. This is always possible by changing the signs of all the coefficients in S, if necesssary.

The lengths of the semi-major (transverse) and the semi-minor (conjugate) axes are respectively

$$\alpha,\ \beta, \tag{6}$$

because $\alpha^2 - \beta^2 = \Delta\sigma/C^2 > 0$.

The equations of the major (transverse) and the minor (conjugate) axes are respectively

$$C(x \sin \theta - y \cos \theta) = G \sin \theta - F \cos \theta, \tag{7.1}$$

$$C(x \cos \theta + y \sin \theta) = G \cos \theta + F \sin \theta. \tag{7.2}$$

The square of the eccentricity $= 2\left[1 - \dfrac{a+b}{\sigma}\right]^{-1}$. (8)

The square of the semi-latus rectum $= \dfrac{8\Delta}{(\sigma - a - b)^3}$. (9)

The foci are

$$x = \frac{G \pm \sqrt{(\Delta\sigma)} \cos \theta}{C}, \; y = \frac{F \pm \sqrt{(\Delta\sigma)} \sin \theta}{C}. \tag{10}$$

The directrices are

$$x \cos \theta + y \sin \theta = \frac{G \cos \theta + F \sin \theta}{C} \pm \frac{\sqrt{(\Delta\sigma)}(\sigma - a - b)}{2C\sigma}. \tag{11}$$

A short proof of Feuerbach's theorem

By E. H. NEVILLE

THE incentre I is the mean centre of loads a, b, c at A, B, C, and therefore

$$\sum (a.NA^2) = 2s.NI^2 + \sum (a.IA^2) = 2s(NI^2 + 2rR),$$

since $s.IA^2 = bc(s-a)$, and $abc = 4R\Delta = 4srR$. The nine-point centre N is the circumcentre of the triangle $D'B'C$, where D is the foot of the perpendicular from A to BC and B', C' are the midpoints of CA, AB, and since this triangle is similar to ACB, the point N is the mean centre of loads $a \cos A, c \cos C$, $b \cos B$ at D, B', C'; hence

$$a \cos A.AD^2 + c \cos C.AB'^2 + b \cos B.AC'^2 = \{\Sigma(a \cos A)\} \times \\ \times (NA^2 + \tfrac{1}{4}R^2),$$

and therefore, since $a.AD = 2\Delta$,

$$\{\Sigma(a \cos A)\}(a.NA^2 + \tfrac{1}{4} aR^2) = 4\Delta^2 \cos A + \tfrac{1}{4} a^2bc$$
$$= 2\Delta(2sr \cos A + \tfrac{1}{2} R^2),$$

But

$$R \sum (a \cos A) = \sum (a.R \cos A) = 2\Delta \text{ and } R \sum \cos A = R + r.$$

Hence

$$2s(NI^2 + 2rR + \tfrac{1}{4} R^2) = 2sr(R + r) + sR^2,$$

that is,

$$NI^2 = (r - \tfrac{1}{2} R)^2.$$

The unexpected detail in this proof is the brevity of the reduction when the triangle $DB'C'$ is used for the determination of NA^2.

A note on Feuerbach's theorem

By K. Srinivasa Rao, *Sri Venkateswara University, Tirupati*

The following Cartesian proof of Feuerbach's theorem, by taking the sides of the concerned triangle as axes and using the well-known properties of the triangle and the circle, appears to be new.

The sides AB, AC of the triangle ABC being taken as the coordinate axes, the midpoints A', B', C' of the sides BC, CA, AB will be $(c/2, b/2)$, $(0, b/2)$, $(c/2, 0)$. Let $I_1(x_1, y_1)$ be the centre of the e-circle opposite A touching the sides BC, CA, AB at P, Q, R respecively. Then

$$x_1 = y_1 \tag{1}$$

and

$$\left.\begin{aligned} AR = s,\ BP = s - c,\ CP = s - b;\\ A'P = (b-c)/2,\ B'Q = (c+a)/2,\ C'R = (a+b)/2. \end{aligned}\right\} \tag{2}$$

By drawing a parallel to AC through I_1 to meet AB and joining I_1R, we find

$$I_1R = s\tan(A/2) = y_1 \sin A. \qquad (3)$$

Let the radical axis of the e-circle and the nine-point circle be

$$lx + my = 1.$$

We know that the difference of the powers with respect to two circles of a point in their plane varies as the distance of the point from their radical axis. It follows as a corollary that the square of the tangent from any point on a circle to another circle bears a constant ratio to its distance from the radical axis of the two circles. From this, since A', B', C' are points on the nine-points circle, $A'P$, $B'Q$, $C'R$ are tangents to the e-circle, and the distances of A', B', C' from $lx + my = 1$ are proportional to $lc + mb - 2$, $mb - 2$, $lc - 2$, we have

$$\frac{(a+b)^2}{lc-2} = \frac{(a+c)^2}{mb-2} = \frac{(b-c)^2}{lc+mb-2}$$

and each

$$= \frac{(a+b)^2 + (a+c)^2 - (b-c)^2}{(lc-2)+(mb-2)-(lc+mb-2)}$$

$$= -(a+b)(a+c).$$

From these,

$$lc - 2 = -(a+b)/(a+c) = -(2s-c)/(2s-b).$$

This gives

$$c(ls-1) = -(s-b)(lc-2).$$

Similarly,

$$b(ms-1) = -(s-c)(mb-2).$$

Hence

$$(ls-1)(ms-1) = (s-b)(s-c)/bc = \sin^2(A/2). \qquad (4)$$

Now, the distance of I_1 from the radical axis $lx + my = 1$ is

$$\frac{(lx_1 + my_1 - 1)\sin A}{(l^2 + m^2 - 2lm\cos A)^{1/2}} = \frac{[(l+m)x_1 \sin A - \sin A]}{(l^2 + m^2 - 2ml\cos A)^{1/2}},$$

by (1) and (2),

$$= \frac{(l+m)s\tan(A/2) - \sin A}{(l^2+m^2-2ml\cos A)^{1/2}}, \qquad \text{by (3)}$$

$$= \frac{s\tan(A/2)\{(1+m) - 2\cos^2(A/2)/s^{-1}\}}{(l^2+m^2-2lm\cos A)^{1/2}}. \quad (5)$$

We observe that,

$$\left[l+m-\frac{2\cos^2(A/2)}{s}\right]^2$$

$$= l^2+m^2+2lm - \frac{4\cos^2(A/2)}{s}\left[(1+m) - \frac{\cos^2(A/2)}{s}\right]$$

$$= (l^2+m^2-2lm\cos A) + (2lm)(1+\cos A) - $$

$$- \frac{4\cos^2(A/2)}{s^2}[s(l+m) - \cos^2(A/2)].$$

But

$$2lm(1+\cos A) - \frac{4\cos^2(A/2)}{s^2}[s(l+m) - \cos^2(A/2)]$$

$$= \frac{4\cos^2(A/2)}{s^2}[s^2(lm) - s(l+m) + \cos^2(A/2)]$$

$$= \frac{4\cos^2(A/2)}{s^2}[(ls-1)(ms-1) - \sin^2(A/2)]$$

$= 0$, by (4).

Hence (5) reduces to $s \tan(A/2)$, showing that the distance of the centre of the e-circle from the radical axis of the e-circle and the nine-point circle is equal to the radius of the e-circle. That is to say, the radical axis touches the e-circle.

Hence the nine-point circle touches the e-circle opposite A. Similarly it touches the other e-circles also.

By changing a to $-a$ and following a similar argument it can be shown that the nine-point circles touches the in-circle also.

I am indebted to Mr. P. S. Rau of Sri Venkateswara University for help in preparing this note.

BOOK REVIEWS

Mathematical Foundations of Quantum Mechanics. By John von Neumann, Translated from the German by Robert T. Beyer, Princeton University Press, 1955, 472 pages, $ 6.00.

THIS is the English translation of the classic work (first published in German in 1932) in which von Neumann shows how the theory of self-adjoint operators in Hilbert space may be used to give an elegant and rigorous formulation of the basic laws of quantum mechanics. In addition to describing this formulation the book presents an exposition of the necessary Hilbert space theory, an elaborate analysis of the physical significance and logical foundations of quantum mechanics and a brief treatment of quantum statistical mechanics. Except for a section on the quantum mechanics of the electromagnetic field and the resulting explanation of the dual wave-particle nature of light the book does not treat the solution of concrete physical problems by quantum mechanics. Thus its overlap with conventional books on quantum mechanics is quite small.

The six chapter headings are : I. Introductory considerations, II. Abstract Hilbert space, III. The quatum statistics, IV. Deductive development of the theory, V. General considerations, VI. The measuring process.

Chapter I begins with some historical remarks culminating in a brief exposition of the original formulations of quantum mechanics due to Schrödinger and Heisenberg respectively. It then passes on to a presentation and critique of Dirac's proof of the equivalence of the two theories by means of his " transformation theory " and concludes with Schrödinger's proof of this equivalence couched in the language of Hilbert space.

Chapter II is a purely mathematical one on the theory of operators in Hilbert space. It contains the author's axiomatic characterization of Hilbert space, a detailed development of the properties of various basic notions and an elaborate analysis of the nature and significance

of the spectral theorem. The spectral theorem is the infinite dimensional generalization of the classical algebraic theorem on the diagonalizability of Hermitian matrices and is the basic tool which makes it possible to avoid the use of the Dirac delta function and to give a mathematically satisfactory treatment of the properties of the observables in quantum mechanics whose associated operators have continuous spectra. In this generalization the role of the Hermitian matrices is played by the "hypermaximal" operators. Hypermaximality for operators is equivalent to another property which is called "self-adjointness" and which is a little easier to describe briefly. The operation of forming the adjoint A^* of a bounded operator A—$(A(\phi), \psi) = (\phi, A^*(\psi))$—may be extended so as to apply to unbounded operators whose domains of definition are only dense in the Hilbert space. The domain of A^* then depends upon that of A but is in general distinct from it. If A and A^* have the same domain and $A(\phi) = A^*(\phi)$ for all ϕ in this domain then A is said to be self-adjoint. The spectral theorem associates to each self-adjoint A a one-parameter family $\lambda \rightarrow E(\lambda)$ of projections (i.e. self-adjoint operators E such that $E^2 = E$) in such a manner that $(A(\phi), \psi) = \int \lambda d\,(E(\lambda)(\phi), \psi)$ for all ϕ and ψ in the domain of A. The family $\lambda \rightarrow E(\lambda)$ is uniquely determined by A and is called the associated resolution of the identity. The full statement of the spectral theorem includes a characterization of the one-parameter families $\lambda \rightarrow E(\lambda)$ which can appear. The reader unfamiliar with the spectral theorem may gain a better appreciation of its meaning by reflecting upon what it reduces to when A happens to have a complete set of eigenvectors. Then $E(\lambda)\,\phi$ is found by expanding ϕ in terms of these eigenvectors and omitting from the sum all terms corresponding to eigenvalues which exceed λ.

In Chapter III the author makes use of the spectral theorem to give a unified and general form to the classical statistical statements of quantum mechanics. Let H be the Hilbert space of all state vectors (wave functions) of a quantum mechanical system. Let ψ be a unit vector in H, let O be an observable and let A be the self-adjoint operator which the theory corresponds to O. Then the probability

measure describing the distribution of values to be expected on measuring O when in the state ψ may be computed as follows from A and ψ. Let $E_A : \lambda \to E_A(\lambda)$ be the resolution of the identity associated with A by the spectral theorem. Then $\lambda \to (E_A(\lambda)(\psi), \psi)$ is a monotone increasing function whose greatest lower bound and least upper bound are 0 and 1 respectively. It is the distribution function for the desired probability measure. More generally, if $O_1, O_2, \ldots, O_n$ are observables (with corresponding operators $A_1, A_2, \ldots, A_n$) such that $E_{A_j}(\lambda)$ commutes with $E_{A_k}(\mu)$ for all j, k, λ and μ, then the probability measure in n-space for the joint distribution of $O_1, O_2, \ldots, O_n$ in the state ψ is determined by the monotone function of n variables $(\lambda_1, \lambda_2, \ldots \lambda_n) \to (E_{A_1}(\lambda_1)E_{A_2}(\lambda_2)\ldots E_{A_n}(\lambda_n)(\psi), \psi)$. In the first five sections of the chapter the meaning and implications of this general statement are examined in considerable detail. In particular there is a discussion of its connection with the concept of "simultaneous measurability" and with "uncertainty principles". The sixth and longest section (over 40 per cent of the chapter) is given over to a detailed presentation of Dirac's theory of the "quantization" of the electromagnetic field. This is included in the present chapter because it is needed in order to show that the rules for computing the probability that an atom will move from one stationary state to another can be derived from the general statistical statement given above.

The general statistical assertion of quantum mechanics formulated in Chapter III was presented more or less dogmatically as a natural generalization and unification of the special cases already known. In Chapter IV it is deduced from axioms. In addition the question of "hidden parameters" is discussed in detail. The basic axiom is that the observables of a system correspond one-to-one to the self-adjoint operators on a Hilbert space in such a way that certain natural regularity conditions are satisfied. Purely mathematical arguments then show that every function of the observables having the properties of a "mean value" must be of the form $A \to \text{Trace}\,(AU)$ where A is the self-adjoint operator corresponding to the observable and U is a non-negative self-adjoint operator. For each unit

vector ϕ the operator U_ϕ where $U_\phi(\theta) = (\theta, \phi)\,\phi$ is non-negative and self-adjoint and leads to the mean value $A \to \text{Trace}\,(AU_\phi) = (A(\phi), \phi)$, which is just the mean value of the observable A in the state ϕ according to Chapter III. When U is not of the form U_ϕ the complex of mean values which it defines is that associated with a statistical mixture of states. Accordingly such a U is said to define a mixture. (In some later literature the term state is extended to cover the mixtures as well and one distinguishes between pure states and mixed states.) The observable associated with the operator A is said to be dispersion-free in the state or mixture defined by U if Trace $(A^2U) = (\text{Trace}\ AU)^2$. In essence von Neumann's argument against the existence of hidden parameters is the following pair of observations : (a) in every state there are observables which are not dispersion-free; (b) no state may be obtained by taking a statistical mixture of other states (or mixtures); i.e. no U is of the form $tU_1 + (1-t)\,U_2$ where $0 < t < 1$ and U_1 is not a scalar multiple of U_2. The balance of the chapter is devoted to discussing certain features of the relationship between states and mixtures. It is shown in particular that when Trace $(U) = 1$ the associated mixture is a statistical mixture of a family of states defined by mutually orthogonal unit vectors.

Chapter V is concerned with thermodynamical aspects of the statistics of quantum mechanics—with the contrast between the measuring process on the one hand which takes states into mixtures and in general increases statistical uncertainty and the process of letting time elapse which takes states into states and does not increase statistical uncertainty. After a full discussion of both processes and their mathematical formulation there is a detailed analysis of the exact degree of statistical uncertainty belonging to a given mixture. This analysis is based upon classical thermodynamics and involves consideration of a " gas " whose molecules are isolated replicas of the system under consideration. It culminates in a formula for the entropy of a mixture. Straightforward calculations based upon this formula then show that any measurement which changes a mixture at all must increase its entropy and that the entropy of an

isolated mixture remains constant in time. The entropy formula is next used to set up the foundations of quantum statistical mechanics. A fundamental contribution of von Neumann to quantum mechanics was his observation that there is a quantum analogue for the notion of a statistical distribution in phase space and that it is the notion of mixture. The mixture of maximum entropy amongst all those giving the expected value of the energy observable a fixed value is a mixture constant in time which is the quantum analogue of the canonical ensemble of Gibbs. The operator for this ensemble is explicitly obtained and some of the basic formulas and notions of classical statistical mechanics are carried over into quantum mechanics. The chapter concludes with a discussion of the paradox produced by the contrast between the constancy in time of the quantum mechanical entropy and the classical principle of entropy increase. A resolution is offered based upon a notion of macroscopic entropy.

In the last chapter the measuring process is further analyzed and shown in a certain sense to be reducible to the process of letting time elapse. Specifically the change produced by a measurement may be regarded as the effect of letting time elapse in the compound system formed from the original system and the observer—the increase in statistical uncertainty being due to lack of knowledge of the state of the observer. The discussion includes a detailed mathematical analysis of the relationship between states and mixtures in the compound system on the one hand and in the component systems on the other.

In spite of the rapid development of both quantum mechanics and the theory of operators in the quarter century since it was written, this book can still be regarded as reasonably up-to-date. Most of the development has been in topics with which the book is not primarily concerned. The fundamental matters with which it deals have undergone relatively little change. Of course various improvements could now be made. Moreover other formulations have been suggested which have advantages (as well as disadvantages) and are preferred by some. Nevertheless a mathematician

at the present time wishing to learn quantum mechanics from someone who speaks his own language could hardly find a better starting point. The book is somewhat less of a "must" for physicists since it has been the fashion to avoid the mathematical subtleties of the spectral theorem and be content with an intuitively plausible, logically unsound, heuristic approach to the difficulties presented by the continuous spectrum. However there is a growing minority of physicists who feel that the time has come to pay more attention to rigor and precise formulation and even the unrepentent anti-rigorist will find interesting discussions of many of the apparent paradoxes of quantum mechanics.

According to the translator's preface the translated manuscript was carefully revised by von Neumann himself. Thus one can assume that it is an accurate translation. The reviewer certainly has no fault to find with it and moreover noted very few misprints.

GEORGE W. MACKEY

Topological Dynamics. By W. H. Gottschalk and G. A. Hedlund, American Math. Soc. Colloquium Publications XXXVI, 1955, pp. viii+152.

A *dynamical system* is such a function $\Pi: X \times T \rightarrow X$ that (i) X is a space, (ii) T is a topological group, (iii) Π is continuous, (iv) $(xt_1)\, t_2 = xt_1\, t_2$ and $xe = x$ for $t_1, t_2 \in T$, $x \in X$ and $e^2 = e \in T$, xt being written for $(x, t)\, \Pi$.

The book consists of two parts, Part II being a collection of examples which will not be adverted to in this review. Part I contains 11 sections, the contents of which will be briefly indicated in due course.

The book is a series of acute and brilliant abstractions in a context of great generality. Historical comments are brief and it is fair to say that, aside from giving a bibliography, the authors are not

concerned with conveying to the reader the reasons for their interest in the material, its position in mathematics or its origins. The bibliography is not extensive.

What are the prerequisites for reading this book ? If the reader is concerned only with being able to comprehend the proofs then he should know the fundamentals of topology of uniform spaces and of topological groups. From time to time he may have to know a little more than generalities in these fields. I do not suppose that many mathematicians are likely to expend the necessary time and energy to get through the proofs merely for the sake of having done so. Hence it might be well to know something about the classical theory of almost periodic functions before starting to read this book. It would be well to have read Chapter XII of Whyburn's Analytic Topology and a couple of Gottschalk's and Hedlund's earlier papers to make the going easier. It is likely that a mathematical physicist, who is more physically than mathematically inclined, would find this book unprofitable without a good deal more background than this. It seems clear that it is written for mathematicians.

Before going into a discussion of the contents, it might be well to tell the story of how Gottschalk became interested in topological dynamics. During the X'mas holidays, going from the University of Virginia to his home he discovered a gap in a proof in Whyburn's book. By the time he returned to the University he had filled the gap and, under Hedlund's influence began to take a deeper look at the context in which he had been working. Gottschalk's first result is Theorem 1 in *BAMS*, vol. 50 (1944), pp. 222-227. Since this is the seed of an interesting feature of the book it is worth examining in detail.

Let $f : X \to X$ be a homeomorphism and say that $x \in X$ is recurrent under f is for each open set U about x there is a positive integer n such that $(x) f^n \in U$. The result asserts that if x is recurrent under f then x is recurrent under f^n for each positive integer n. The proof is by induction and the familiar box-argument (if $n + 1$ things are contained in n boxes then one box at least contains two things). Now

let T be all the integers and define $\Pi : X \times T \to X$ by $(x, t)\, \Pi = (x) f^t$. We have before us a dynamical system and the question is : How can we formulate and prove for a general dynamical system an analog of Gottschalk's result ? I would strongly urge the reader to spend a while in trying to answer this question in his own way so that he may the more appreciate Gottschalk and Hedlund's answer.

§ 1. Notation, terminology and about 30 definitions.

§ 2. Relatively dense sets, which lie at the basis of the classical theory of almost periodic functions are generalized as follows : A subset A of T is left (right) syndetic if $T = AK\,(Y = KA)$ for some compact set K. An orbit-closure is a set $(xT)^*$, $x \in X$, where * means closure. If X is compact there is at least one miminal orbit-closure. If A covers X then A is star-closed if for each closed $Y \subset X$ the union of all members of A meeting Y is closed. Here is an example of the transfer of a property of T to one of its syndetic subsets : If X is locally connected, T_1 and normal, if the orbit-closures under T form a star-closed partition and if S is a syndetic invariant subgroup of T then the orbit-closures under S form a star closed partition of X.

§ 3. Recursion. If $x \in X$ then the period of x is the largest subset P of T with $xP = x$. It is clear that P is a subgroup of T. T is periodic at x if P is syndetic. Let X be the set of all continuous functions on T to T, the reals. Then sin $\in X$ and the set of all t such that for each t_0, (t_0) (sin t) = the sine of $(t_0 + t)$ is the group G of integral multiples of 2Π. This group is syndetic since $T = G + [0, 2\,\pi]$ and $[0, 2\pi]$ is compact. Hence sin is periodic in the new sense. (It is sometimes difficult to adhere to the authors' notational conventions.) Fix a class a of ("admissible") subsets of T and say that $x \in X$ is *recursive* if for each open set U about x we have $xA \subset U$ for some $A \in a$. The authors define nine variants of this and include a table of 13 implications. The inheritance theorem brings us to the question to which we alluded earlier : *For each subgroup G of T let there be distinguished in G certain subsets, called G-admissible, such that :*

(I) *If $A, B, C \subset T$ such that $A \subset BC$, if A is T-admissible and if C is compact then B is T-admissible.*

(II) *If G is a closed syndetic subgroup of T and if $A \subset G$, then A is G-admissible if and only if A is T-admissible. Let T be locally compact and let S be a closed invariant syndetic subgroup of T. Then*

(1) *if $x \in X$, then S is recursive at x if and only if T is recursive at x.*

(2) *S is pointwise recursive if and only if T is pointwise recursive.*

An ingenious way of saying that a subset of a group " accumulates an infinity " is to say that it is " extensive ". A set in T is replete if it contains some bilateral translate of each compact set in T; it is extensive if it meets every replete semigroup. (In the notes on p. 29, (3.38) should be (3.37)).

§ 4. This section, on almost periodicity, is perhaps the most enlightening in the book. To say that T is almost periodic at x (or that x is almost periodic under T) means that T is recursive at x with " admissible " set meaning " left syndetic " set. Here one finds the relations between almost periodicity, compactness of the orbit and discrete almost periodicity. It should be noted that eight types of almost periodicity are employed. In this section also one finds the theorem that a compact minimal orbit is actually a group and the theorem on the existence of a mean for a function on T to X. (In (4.87) the authors failed to note B. Eckmann's contribution.)

§ 5. Regular almost periodicity. This section contains its quota of inheritance theorems and, among other things, the result that an almost periodic homeomorphism on a compact space is the limit of the powers of a regular almost periodic homeomorphism. (Presumably, $(x\phi, x\psi)\rho$ should be written for $\rho(x\phi, x\psi)$).

§ 6. A consideration of the properties of replete semi-groups and applications. In particular : If M is a closed subset of X, if P is a replete semigroup in T and if $x \in M - MP$ then P contains a replete semigroup Q such that x is not in $(xQ)^*$.

§ 7. The first inheritance theorem here is the solution of the problem we posed earlier for the reader. *If S is a closed syndetic*

subgroup of T then S is recurrent at x if and only if T is recurrent at x. We recall that T is recurrent at x means : If U is an open set about x then there is an extensive set A such that $xA \subset U$. We leave to the reader the task of reconciling this and the earlier use of recurrent. There are, of course, other kinds of recurrence treated in this section, including the non-wandering points of Birkhoff.

§ 8. Incompressibility. This notion is best indicated by stating a simple result. If $f : X \rightarrow X$ is a homeomorphism and if X is compact then f is recurrent (at each point of x) if and only if, for each closed set $A \subset X$, $(A)f \subset A$ implies $(A)f = A$. In other words, closed subsets are incompressible under recurrent homeomorphisms. There is a dual result (presumably not known to the authors) as follows : If the compact semigroup S acts on the compact space X, if $t \in S$ and if A is a closed subset of X such that $A \subset At$ then $A = At$. In fact the semigroup $T = \{t^n \mid n \geqslant 1\}^*$ *acts* as a group of homeomorphisms on A and in particular if e is the unit of the maximal subgroup of T then $A = Ae$.

§ 9 and § 10 are devoted to transitivity and asymptoticity while § 11 is a collection of (mostly known) results on function spaces.

A. D. WALLACE

Thinking by machines: a study of Cybernetics. By Pierre de Latil, Translated from the French by Y. M. Golla, Sidgwick & Jackson, London, 1956, sh. 45.

THIS book is a good example of the difficulties one runs into in trying to write on Cybernetics. Its plan is ambitious and its scope very wide. In one connection or another de Latil touches on a variety of topics from servo-mechanisms to cosmology and from internal combustion engines to oriental mysticism. He offers comments on the behavior of glaciers, of plants, of animals, on meteorology, quantum mechanics, psychology, neuro-anatomy, on continuous creation of matter, on practically everything one can think of.

Consequently it is never clear what class of readers the book is aimed at. The style is very uneven and at its worst comes perilously close to the sensationalism of the 'sunday press.' This is unfortunate since de Latil himself seems to be well aware of these pitfalls and frequently criticises vigorously the attempts to misrepresent facts of science by popularization and over-simplification. de Latil's self-imposed restriction to eschew 'mathematical technicalities' often forces him to complicate with a mass of verbiage what are essentially simple concepts. As a typical example one can cite his definition of probability which he claims "accords with logical thought better than does the classical definition": 'The probability of an effect is the ratio of that part of its field of causal function, which falls within the frame of probability of the variable of this function, to the frame of probability'.(p.175)

Thinking by machines is actually the first part of a two volume study. The sequel, to be called *L'Homme en Equation* will, according to de Latil, deal with the 'human machine' and the details of construction of *Machina Liberata* which will function as a simulator, in some sense, of the former. But here de Latil is mostly concerned with the 'artificial machines' and their classification in a cybernetic hierarchy. A machine of the first degree is one that accomplishes a single simple act. A typewriter key, for example, or a hand-brake. A second degree machine, like the 4-stroke internal combustion engine, performs a fixed sequence of simple acts. A third degree machine incorporates a detector device, in addition, so that the fixed sequence of operations takes place only if some external condition is satisfied. A typical example is the automatic fire alarm. So much for the 'classical machines'.

In Chapter 3, de Latil starts discussing the truly 'cybernetic machines' which because of feedback are able to control and stabilize external phenomena. His descriptions of the industrial control systems in Chapters 3 to 6 and his discussion, in Chapter 10, of the use of machines in the study of animal behaviour, are perhaps the best parts of the book. In the fourth degree are classified all those regulators which like the Watt governor maintain themselves

in a fixed equilibrium state. A fifth degree machine is one that is capable of changing its internal setup so that it can maintain itself in one of a set of predetermined equilibrium states. Ashby's Homeostat is probably the only one of this type available now, although other similar machines can be easily visualized. A sixth degree machine would presumably go a step further and choose its own set of equilibrium states and adapt its 'internal programme' accordingly. The seventh degree machines, according to de Latil, have cosmological and evolutionary implications, but at this stage his discussion becomes highly speculative and nebulous.

de Latil's excessive preoccupation with essentialist definition sometimes obliges him to subscribe to odd points of view. He classifies electronic calculators as second degree machines while the automatic volume control of the radio receiver as a fourth degree one ! Talking of calculating machines he writes, "They do not reach into the domain of the new science; their behaviour is in fact determined with complete rigidity and is not organised by a network of clinamens (feedback mechanisms). They do not belong to the realm of Cybernetics which is that of super-automata." But if this is true, it can only be because de Latil considers the programme as being something external to the computer. That in ordinary usage one stores the programmes as punched tapes outside the calculator has no significance apart from that of convenience. One might as well have translated all the programme in terms of hardware and made it part of the machine. In fact, in some large scale machines with extensive drum storages programmes do remain stored in the drum all the time. Now, it is well known that extremely generalized programmes can be written that modify themselves on the basis of subsequent events. Shannon, in fact, has suggested a programme to get a computer to play chess and improve on its strategy on the basis of its past performance. A computer with such a programme incorporated in it and supplied with suitable end-organs should function as well as, if not better than, de Latil's fifth degree machine.

His desire to give credit to the continental workers (mostly French) often leads him to do less than justice to facts. His statement,

" Today the best exclusively binary calculating machines exist in England ", could not have been true even in 1953 (the date ascribed to the French edition). There were half a dozen large scale binary computers actively being used in the United States at that time, including the machines at Princeton, M.I.T., Illinois and the Ballistic Research Laboratory. Concerning Charles Babbage, de Latil merely notes, he " attempted the construction of an analytic machine governed by a perforated ' programme ' belt " and that having " worked at it for twenty-five years...in the end died without having accomplished his project. " Von Neumann's contributions to the field are barely mentioned ; nor is anything said about his work on ' probabilistic logic '. While discussing the use of binary calculators in solving problems of deductive logic, de Latil does not refer to the pioneering work of Shannon. In fact in all the 300 and odd pages of the book, Shannon's name is not mentioned even once !

Thinking by machines is in many respects a disappointing book. The pity is all the greater since obviously a good deal of effort has gone in the making of it. Perhaps this merely serves to underscore the fact that what is needed now in Cybernetics is less of exposition and more of research with patient attention to details.

The translation from the French by Y. M. Golla has been extremely well done and the printing and the general getup of the book are quite satisfactory.

R. NARASIMHAN

Introduction to abstract algebra. By C. Racine, Published by S. Viswanathan, Madras, 1957, pp. xii+194, Rs. 6.00.

THIS is intended to be a text book for the honours and post-graduate students of Indian Universities.

There is an introduction in which the author briefly traces the growth of algebra starting with the ancient mathematics of the Egyptians and the Babylonians, leading up to the present day mathematics of Bourbaki.

The following is a brief account of the contents of the book. The first chapter introduces groups and deals with some of their elementary properties. The structure theorem for finite abelian groups is included. The second chapter is devoted to rings. Among the topics included are the elementary theory of ideals and the theory of factorization of an element into irreducible elements. Polynomial rings are defined and some theorems on the extension of properties of the ground ring to the polynomial ring are proved. Chapter III gives an account of field-theory. The existence and uniqueness of the algebraic closure is assumed without proof. Galois theory of finite extensions is studied, and applications to the theory of equations are given. There is also a section on ordered and formally real fields. The fourth and the final chapter deals with linear algebra and matrices. The problems of reduction to canonical forms are studied and the structure theorem of finitely generated modules over a Euclidean ring is proved using the elementary divisors theorem.

In the opinion of the reviewer, some portions of the section on divisibility are not clearly written. The author does not define explicitly the notions of a unique factorization ring and factorization ring, though, of course, he uses both the terms. Theorem X on p. 69 is rather misleading. It states, " In a principal ideal ring, every factorizable element has a unique factorization, " whereas it is true that every non-unit in a principal ideal ring has a unique factorization. This important theorem has not been explicitly stated in the text. (However, in the supplementary list of " Additions and Errata ", the author adds this statement as a remark).

In defining the polynomial ring in one variable over a ring with unit (or in a remark that follows) the author prefers to leave the 'indeterminate' undefined. The reviewer feels that the notion of the indeterminate could have been explained better.

Vector spaces are defined in a section of Chapter III. In view of their importance, the reviewer feels that they could have been given more prominence.

In the chapter on matrices, the section dealing with the one-one correspondence between matrices and linear transformations could have been brought earlier.

The errata list given at the end of the book and the supplementary one which the author has issued do not exhaust the misprints.

In spite of the above criticisms, there is no doubt that this book is a welcome addition to the already existing text books on algebra. The price of the book is quite low and the student community should all the more be thankful for that. The reviewer feels, as the author hopes in the Introduction, that this book will contribute to create in the student a lasting enthusiasm for mathematics.

R. SRIDHARAN

Elementary theory of numbers. By Harriet Griffin, McGraw Hill Book Company, Inc. (1954), pp. 204, $ 5.00.

This book meant for the undergraduate or graduate student of the American Universities covers such elementary topics in number theory as diophantine equations, congruences, quadratic residues and has also a chapter on polynomials and another on partitions. The development of the topics in all chapters is very smooth, for example, from the discussion of the solution of the linear diophantine equations to the notion of the greatest common divisor and coprimality. Chapter 7 could have been better understood if the author had introduced the notion of a group and translated the questions considered here in terms of finite groups. The chapter on quadratic residues is concise and also complete. The book throughout keeps the spirit of number theory alive and one is often enlivened by the brief survey of the historic background of the topics.

But, at some places, there are noticeable omissions, for example, rational numbers and real numbers have not at all been defined. In Chapter 2, the author defines a polynomial as a " rational integral algebraic expression ", the meaning of which is in no way clear and the author builds her chapter on polynomials on this

definition. The book is very well got up and there are practically no printer's errors.

S. S. R.

Ordinary level applied mathematics. By J. E. C. Gliddon, University Tutorial Press Ltd. London, pp. 491, 1956.

THE book has been written to meet the needs of candidates offering mechanics and hydrostatics at the ordinary level of the General Certificate of Education, England. In terms of the standards in India, the book roughly covers up the syllabus prescribed in statics, dynamics and hydrostatics for the intermediate science examination conducted by various Boards of Higher Secondary Education in the country.

The subject matter is generally the same as would be found in any book of this standard, but the author has made the treatment distinct from most of them by introducing some new features. (i) The author has made the subject matter interesting by including in it topics like the equilibrium of an aeroplane in steady level flight, jet propulsion, vertical structure of the atmosphere, on the basis of latest observations. (ii) The concept of vectors and the operations of vector addition and subtraction are introduced in the early part of the book in a natural manner. These concepts have been subsequently used freely. This has enabled the author, among other things, to emphasize that the equation of motion and impulse-momentum equations are in fact vector equations. The topic of relative velocities becomes elegant with the use of vectors. (iii) Every time a new concept is introduced, the author has given the experimental basis for it. This, besides making the concept easily intelligible convinces the reader that what he is reading is not only of academical interest but is of everyday application. (iv) The solved examples and exercises set for the students are almost all numerical. Their correct solution not only requires the precise appreciation of the units and dimensions of the physical quantities, but also the thorough grasp of the fundamental principles, for looking at the numerical result the

student cannot work backwards to help his thinking. However, the reviewer would have appreciated it very much if among these numerical exercises some analytic exercises had also been given. (v) The choice of the examples is judicious and interesting. Most of the problems relate to our everyday experience. (vi) The concept of mass is usually difficult to grasp at the first reading. Consequently, it has been introduced in Chapter 18, assuming that by the time the reader comes to this chapter, he is sufficiently mature to assimilate it. It is a pleasure to note that the author has set up the concept of mass in an elegant and accurate manner.

In the opinion of the reviewer the book is ably written. The treatment is elegant, simple and yet rigorous. The book will be found extremely useful for the students reading for intermediate science and for the first degree in engineering of the Indian Universities.

P. L. BHATNAGAR

Cycles of recurring decimals, Vol. II. By D. R. Kaprekar, Published by the author, Khare Wada, Deolali, pp. iv + 47. Price Rs. 6.00.

VOLUME I of this book was reviewed in the Mathematics Student, 18, 1950, p. 165. The present volume is an extension of the first and deals with cycles of recurring decimals for fractions of the form m/n, $163 \geqslant n \geqslant 213$. Except for the change of nomenclature from minumetor to minum, there is no change in the plan. The author gives also a method for finding the period for $1/N^2$ from that for $1/N$. The peculiar structure of the period numbers of numbers like 89991 is considered and factors of some of the period numbers are given. Numbers of cycles for reciprocals of primes upto 13709 are also given. Of what particular importance is the prime 13709 is not clear to the reviewer.

The author has taken pains not only over the calculations but also in correcting some misprints which could have been easily avoided.

H. GUPTA

Journal of science and engineering research, Volume I, Part 1. Published by the Indian Institute of Technology, Kharagpur, India, 1957.

THIS is the opening volume of a new journal ' devoted to the publication of original works in all branches of pure and applied sciences including the fields of engineering and technology'. Each volume of the journal is to appear in two parts and is priced at Rs. 10 (15 sh. or $2½). The present issue, running to some 150 pages, includes titles like the following: ' On the hysteresies of tube oscillator ', ' On certain boundary problems of the theory of elasticity ', ' Determination of mineral proportion in Indian coal : an application of X-ray diffraction method ', ' Screw-pump and its application ', ' The evaluation of the co-volume function in Goldies' method of internal ballistics '.

Tables of integral transforms, Vol. II. Edited by A. Erdèlyi, California Institute of Technology, MacGraw-Hill, 1954, pp. 452, $8.00.

NEWS AND NOTICES

G. C. Nivas, Lecturer, Victoria College, Gwalior has been admitted as a life-member of the Society.

The following persons have been admitted to membership of the Society : Bhagwan Das Agrawal, P. S. Ananthakrishnan, Mrs. M. Apte, S. G. Arsekar, L. R. Bharadwaj, M. Bhaskaran, S. R. Bisi, M. C. Chakrabarty, J. R. Choksi, J. L. Desai, Teeka Rao Desai, S. H. Dwivedi, B. N. Ganguli, Girdhari Lal, Miss Jaya Devi, R. K. Jindia, V. L. Kantam, A. Krishnamurti, V. Krishnamurti, S. C. Mittal, Madan Mohan Lal, M. C. Narasimhan, Mrs. Patel, M. S. Rajagopalan, Sampuran Singh, S. Sarwal, K. B. Shah, P. S. Subramanian, M. Tikku, P. Tiwari, H. K. Varma, N. Yegyanarayanan.

The following members of the American Mathematical Society have been admitted as members under the reciprocity agreement : J. Andrushkiw, George Bachman, Oscar R. E. Barker, Robert G. Bartle, E. F. Beckenbach, P. R. Beesack, Rev. S. J. Bezuszka, B. H. Bissinger, R. L. Blair, J. H. Blau, A. W. Bodyreff, John W. Brace, Glen Bredon, A. B. Brown, H. D. Brunk, R. G. Buschman, Ward Cheney, Randolph Church, William J. Cody Jr., Eckford Cohen, Hirsh G. Cohen, W. A. Cowan, Myrle V. Cross, Arthur E. Danese, Robert J. Dickson, K. O. Friedrichs, Adolph W. Goodman, George R. Grainger, D. S. Greenstein, Theodore Hailperin, J. S. Hanna, G. A. Hedlund, N. Knol Hersh, R. G. Heyneman, Harry Hochstadt, John Isbell, T. R. Jenkins, William E. Jenner, R. B. Johnson, Kenneth E. Kain, Naoki Kimura, W. J. Klinczak, Sam G. Kneale, Paul J. Koosis, R. A. Kunze, Herbert R. Leifer, Eugene Leimanis, Frank Levin, Lillian R. Lieber, Clarence I. Lubin, Paul J. McCarthy, Robert R. MaCready, Eugene K. McLachlan, Holbrook M. McNeille, Morris Marden, May H. Maria, R. B. Merkel, Eugene D. Nichols, C. D. Olds, M. W. Oliphant, Henry J. Osner, Hiram Paley, C. D. Papakyriakopoulos, Tullio G. Pignani, Pasquale Porcelli, R. E. Priest, Alfred K. Putnam, Earl D. Rainville, C. B. Read, Thomas R. Richards, Theodore J. Rivlin, J. B. Roberts,

S. C. Saunders, Joseph A. Schatz, A. L. Shields, Robert J. Silverman, Abe Sklar, Bernard Solmer, John D. Thomas, M. L. Tomber, Sister M. Denis Treacy, Robert M. Thrall, Ernest W. Trost, Howard G. Tucker, L. H. Tulloch, E. F. Tyler, William Wernick, Sonia S. Wohl.

The Council of the Society has been reconstituted. The members of the Session 1957-1959 are : Professor V. Ganapathy Iyer (*President*), Professor S. Mahadevan (*Secretary*), Professor G. L. Chandratreya (*Treasurer*), Professor C. T. Rajagopal (*Librarian*), Professor K. Chandrasekharan (*Editor*), Professor P. L. Bhatnagar, Professor B. Madhava Rao, Professor S. Minakshisundaram, Dr. N. S. Nagendranath, Dr. A. Narasinga Rao, Professor V. V. Narlikar, Professor Ram Behari, Professor N. R. Sen, Dr. N. G. Shabde and Professor S. M. Shah.

We regret to announce the death of Sri C. Bhaskarayya, a life-member of the Society.

Dr. R. S. Mishra of the University of Delhi has proceeded to U.S.A. for further study.

Dr. R. P. Bambah has been appointed visiting professor at the University of Notre Dame for one year from September 1957.

Mr. S. Parameswaran of Kerala University has gone for higher studies at the University of Illinois.

Dr. S. Swaminathan has proceeded to Paris for further study.

The following seven members of the School of Mathematics of the Tata Institute of Fundamental Research have been awarded foreign fellowships for higher research: Mr. M. S. Narasimhan, Mr. C. S. Seshadri and Mr. B. V. Singbal by the Centre National de Recherche Scientifique, Paris ; Mr. P. K. Raman by the French Government ; Mr. S. S. Rangachari by the University of Göttingen; and Mr. K. Varadarajan and Mr. R. Sridharan by Columbia University, New York.

Mr. V. Venugopal Rao of the Tata Institute of Fundamental Research has been awarded a Ph. D. degree by the University of Bombay for his thesis entitled 'The lattice point problem for indefinite quadratic forms'.

Mr. C. S. Seshadri of the Tata Institute of Fundamental Research has been awarded a Ph. D. degree by the University of Bombay for his thesis entitled 'Generalized multiplicative meromorphic functions on a complex analytic manifold'.

A Conference on Analytic Functions was held at the Institute for Advanced Study, Princeton, N. J., U. S. A. on September 1-14, 1957, under the sponsorship of the Air Force Office of Scientific Research. The following mathematicians attended it : L. V. Ahlfors (U. S.), R. F. Arens (U. S.), H. Behnke (W. Germany), S. Bergman (U. S.), L. Bers (U. S.), A. Beurling (U. S.), G. Bjorck (Sweden), S. Bochner (U. S.), R. Bott (U. S.), H. J. Bremermann (U. S.), C. R. Buck (U. S.), L. Carleson(Sweden), K. Chandrasekharan (India), J. Deny (France), G. de Rham (Switzerland), L. Ehrenpreis (U. S.), K. Fan (U. S.), L. Foures (France), A. M. Gleason (U. S.), H. W. J. Grauert (W. Germany), R. C. Gunning (U. S), W. K. Hayman (U. K.), M. Heins (U. S.), M. Hervé (France), F. Hirzebruch (W. Germany), S. Hitotumatu (Japan), K. Hoffman (U. S.), J. A. Jenkins (U. S.), S. Kakutani (U. S.), I. Kaplansky (U. S.), K. Kodaira (U. S.), M. Koecher (W. Germany), J. Lehner (U. S.), P. Lelong (France), J. Leray (France), C. Loewner (U. S.), G. W. Mackey (U. S.), W. T. Martin (U. S.), M. Morse (U. S.), Z. Nehari (U. S.), R. Nevanlinna (Finland), L. Nirenberg (U. S.), M. Ohtsuka (Japan), M. Parreau (France), H. Petersson (W. Germany), A. Pfluger (Switzerland), H. Rademacher (U. S.), H. E. Rauch (U. S.), R. Remmert (W. Germany), P. C. Rosenbloom (U. S.), M. L. Royden (U. S.), L. Sario (U. S.), A. Selberg (U. S.), D. C. Spencer (U. S.), G. Springer (U. S.), W. F. Stoll (W. Germany), L. Waelbrock (Belgium), J. L. Walsh (U. S.), J. Wermer (U. S.), F. Wolf (U. S.), A. Zygmund (U. S.).

The programme was prepared by a Committee consisting of Professor Marston Morse (Chairman), Professor Arne Beurling and

Professor Atle Selberg. The Conference was considered to be successful beyond all expectations. The principal addresses will appear in a book to be edited by the Conference Committee and published by the Princeton University Press.

All the unpublished mathematical manuscripts of the late Srinivasa Ramanujan have been published, in two volumes, by the Tata Institute of Fundamental Research, with the financial assistance of the Sir Dorabji Tata Trust.

With effect from January 1, 1958 Professor S. M. Shah has taken over as editor of *The Journal of the Indian Mathematical Society* and *The Mathematics Student.* All editorial correspondence may hereafter be addressed to Professor S. M. Shah, Muslim University, Aligarh.

THE INDIAN MATHEMATICAL SOCIETY

Statements of Account for the years ended 31st March 1956 and 31st March 1957

Receipt Account for the year ending 31-3-1956

RECEIPTS

	Rs.	a.	p.	Rs.	a.	p.
To **Life Composition Fee*, membership and subscription fees**				7,527	13	6
„ **Sale of Publications**				1,493	14	2
„ **Grants-in-Aid**						
Bombay University : 54-55, 55-56	400	0	0			
Madras University : 55-56	150	0	0			
Osmania University	100	0	0			
Ministry of National Resources and Scientific Research, Central Government	2,000	0	0			
Tata Institute of Fundamental Research ...	1,000	0	0			
Honorary Secretary, National Institute of Science	1,000	0	0			
				4,650	0	0
„ **Interest on Bank Balances**				30	11	9
„ **Interest on Investments**				420	0	0
				14,122	7	5
„ **Provision for Expenses**				1,145	0	0
„ **Opening Balances**						
Cash on hand	*Nil*					
Balance in C/A with the Indian Bank Ltd. Madras	5,315	12	3			
Balance in C/A with the Bank of Mysore Ltd., Bangalore	6,093	13	3			
Fixed Deposit with the Indian Bank Ltd., Mylapore, Madras	12,000	0	0			
Savings Bank A/C with the Indian Bank Ltd., Mylapore, Madras	890	15	1			
National Savings Certificates of the face value of Rs. 9,000/-	9,000	0	0			
Dr. A. Narasinga Rao Gold Medal Fund (in the form of N. S. Cr.)	1,000	0	0			
Advance with Dr. Chandrasekharan	487	5	3			
				34,787	13	10
Total Rs.				50,055	5	3

*Life Composition Fee is Rs. 1,900/-.

Bombay,

Dated : 7th May, 1956.

MATHEMATICAL SOCIETY

Payment Account for the year ending 31-3-1956

PAYMENTS

	Rs.	a.	p.	Rs.	a.	p.
By **Printing of Journals**				10,170	9	0
„ **Management Expenses**						
Printing & Stationery	204	15	9			
Postage of Journals	568	15	0			
Office Postage	348	10	6			
Office Expenses	394	14	3			
Honorarium to Treasurer's clerk	190	0	0			
Audit Fees	25	0	0			
Bank Commission	40	11	0			
Travelling of the Hon. Secretary	190	8	0			
Travelling of Prof. Eichler	76	0	0			
Subscription to Journals	426	15	3			
Book binding charges	385	8	3			
Dr. A. Narasinga Rao Gold Medal Expenses	102	10	0			
				2,954	12	0
„ **Library Books**				3,116	12	3
„ **Furniture and Dead Stock**				373	4	0
				16,615	5	3
„ **Closing Balances**						
Cash on hand with the Hon. Treasurer ...	20	13	9			
Balance in C/A with the Indian Bank Ltd., Madras...	4,744	9	11			
Balance in C/A with the Sangli Bank Ltd., Sangli	4,996	6	6			
Fixed Deposit with the Indian Bank Ltd., Madras...	12,420	0	0			
Balance in S/B Account with the Indian Bank Ltd., Madras	890	15	1			
National Savings Certificate (General Account of the Face Value of Rs. 9,000/-)	9,000	0	0			
Dr. A. Narasinga Rao Gold Medal Fund (in the form of N.S.Cr.)	1,000	0	0			
Advance with Dr. K. Chandrasekharan ...	367	2	9			
				33,440	0	0
Total Rs.				50,055	5	3

As per our report of even date.

P. G. BHAGWAT

Chartered Accountants.

Jiji House,
Bombay-1.

THE INDIAN

Receipts and Payments Account for

RECEIPTS

	Rs.	a.	p.	Rs.	a.	p.
To **Life Composition fees**				400	0	0
,, **Membership & Subscriptions**				5,174	3	6
,, **Subscriptions from Associate Societies**				345	0	0
,, **Sale of Publications**				557	0	0
,, **Grant-in-aid**						
Bombay University	200	0	0			
Madras University	150	0	0			
Osmania University	100	0	0			
Ministry of Natural Resources and Scientific Research, Ministry of Economics, Government of India, New Delhi	2,000	0	0			
Tata Institute of Fundamental Research, Bombay	1,000	0	0			
National Institute of Sciences of India, Delhi	2,000	0	0			
				5,450	0	0
,, **Interest on Bank Balance**				64	4	7
,, **Interest on F.D. with Indian Bank, Madras** ...				434	11	2
Total Receipts Rs.				12,425	3	3
To **Opening Balance**						
Cash on hand	20	13	9			
Current A/c. balance with Indian Bank Ltd., Madras	4,744	9	11			
F.D. with the said Bank	12,420	0	0			
Savings A/c. balance with the said Bank ...	890	15	1			
National S. Certificates	9,000	0	0			
Dr. A. Narsinga Rao Gold Medal Fund in the form of N.S.Cr.	1,000	0	0			
Balance in Current A/c. with Sangli Bank ...	4,996	6	6			
Advance with the Editor of Journal	367	2	9			
				33,440	0	0
Total Rs.				45,865	3	3

Bombay,

Date : 10th October, 1957.

MATHEMATICAL SOCIETY

the Year ending 31st March 1957

PAYMENTS

	Rs.	a.	p.	Rs.	a.	p.
By Printing of Journals				10,724	9	9
,, **Management Expenses**						
Printing	617	7	0			
Treasurer's Office expenses	50	0	0			
Stationery	21	12	6			
Bank Commission	19	13	0			
Honorarium to Editor's Secretary and Treasurer's Clerk	495	0	0			
Postage for Journals	1,808	0	0			
Travelling Expenses	171	0	0			
				3,183	0	6
,, **Subscriptions to Journals**				423	0	3
,, **Book-binding charges**				340	8	0
,, **Dr. A. Narasinga Rao Gold Medal Expenses** ...				100	1	0
,, **Library books**				2,300	9	4
,, **Furniture and Dead Stock**				295	4	0
,, **Last year's outstanding expenses paid during the year**				1,145	0	0
Total Payments Rs.				18,512	0	10
By **Closing Balance**						
Cash on hand with the Treasurer	1	8	3			
Current Account Balances—						
Indian Bank, Madras	645	10	6			
Sangli Bank	2,428	13	10			
Savings Bank Balances—						
Indian Bank, Madras	926	13	10			
Fixed Deposit with the Indian Bank Ltd., Madras	12,854	11	2			
National S. Certificate F.V.	9,000	0	0			
Dr. A. Narsinga Rao Gold Medal Fund in the form of N.S.C. (F.V.)	1,000	0	0			
Advance with Oxford University Press ...	10	6	5			
Advance with the Editor of Journal	485	0	6			
				27,353	2	5
Total Rs.				45,865	3	3

Examined and found correct.

P. G. BHAGWAT,

Chartered Accountants.